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General Residual Power Series Method: Explicit Coefficient Derivation and Unified Laplace-like Transform Approach for Fractional PDEs

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Abstract

This work introduces the General Residual Power Series Method (GRPSM) as a unified analytical framework encompassing the conventional Residual Power Series Method (RPSM) and its Laplace-like transform variants. By deriving a universal coefficient formula, the GRPSM clarifies the recursive structure of residual-based series solutions and removes the need for repeated limit evaluations across different transform formulations. It is shown that all Laplace-like RPSM variants yield identical coefficient recursions, indicating that their differences stem only from algebraic reparametrizations of the same underlying mechanism. This analytical invariance reveals that the classical RPSM already represents the simplest and most direct form of the unified approach, providing a clear theoretical basis for transform-based extensions in time-fractional and related differential equations.

Keywords: fractional differential equations; series solutions; residual power series method; general Laplace-like transforms; transform invariance

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1. Introduction

The Residual Power Series Method (RPSM) has emerged as a powerful and straightforward analytical tool for constructing approximate solutions of nonlinear and fractional differential equations. Originally proposed by Abu Arqub [1], the method expresses solutions as fractional power series expansions and has gained wide attention due to its simplicity, convergence properties, and flexibility for different classes of problems.

To enhance its applicability, various extensions have been developed by combining the RPSM with integral transforms. Eriat et al. [2] introduced the Laplace Residual Power Series Method (LRPSM), which merges the Laplace transform with the RPSM framework to generate solutions directly in the Laplace domain. Subsequent studies applied similar strategies with other Laplace-like transforms, such as the Elzaki, Aboodh, Sumudu, Kamal, and Pourreza transforms, enabling analytical representations in diverse transform spaces [3–7]. These variants share the same conceptual foundation—constructing a power series solution in the transform domain—but differ in their kernel functions and algebraic formulation.

The integration of these transform-based RPSM variants culminated in a broader theoretical framework known as the General Residual Power Series Method (GRPSM) [8,9]. The GRPSM offers a unified platform that encompasses all Laplace-like transforms while preserving the simplicity of the original RPSM. However, despite its generality, two essential questions have remained open: (1) Are the coefficients obtained from different Laplace-like transforms mathematically consistent? (2) How can a rigorous analytic relation that proves this equivalence be established?

Despite the success of traditional RPSM with various Laplace-like transforms in constructing series solutions for time-fractional differential equations, a significant challenge remains. The existing methods often involve computationally complex calculations of series solution coefficients at each step of the solution process. Furthermore, a critical question regarding the consistency of coefficients obtained through the RPSM with various types of Laplace-like transforms remains unanswered. Our recent work [9] addresses these limitations by deriving explicit formulas for the coefficients in the LRPSM. However, this present article not only simplifies computations but also ensures consistency across various Laplace-like transforms, enhancing the efficiency and reliability of the RPSM. This not only streamlines the solution process but also offers a universally applicable approach for RPSM with Laplace-like transform variants.

The paper is organized as follows. Section 2 presents preliminary definitions and lemmas concerning fractional derivatives and the general integral transform. Section 3 establishes the GRPSM framework, derives the universal coefficient relation, and proves its transform invariance. Section 4 illustrates the theoretical findings through representative time-fractional PDE examples.

2. Preliminaries

This section summarizes the basic definitions and properties of fractional calculus and general Laplace-like integral transforms, which form the analytical basis of the proposed GRPSM. All functions are assumed to be sufficiently smooth and of exponential order.

2.1. Fractional Calculus Preliminaries

Definition 1 (Caputo time-fractional derivative)**.** *Let $\psi(x,t)$ be continuously differentiable with respect to t for $t>0$. The Caputo fractional derivative of order $\alpha>0$ is defined as*

$$D_t^{\alpha}\psi(x,t) = I_t^{n-\alpha}\frac{\partial^n\psi(x,t)}{\partial t^n} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1}\frac{\partial^n\psi(x,\xi)}{\partial\xi^n}\,d\xi, \tag{1}$$

where $n-1<\alpha\leq n$ and I_t^{β} denotes the Riemann–Liouville fractional integral operator

$$I_t^{\beta}\psi(x,t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\xi)^{\beta-1}\psi(x,\xi)\,d\xi, \qquad \beta>0.$$

Remark 1. *For the repeated application of the Caputo derivative used throughout this work, we assume the standard regularity condition*

$$u(x,t)\in C^M([0,T]), \qquad \text{for some integer } M\geq\lceil(n+1)\alpha\rceil,$$

so that all Caputo derivatives up to order M exist, are continuous on $[0,T]$, and remain bounded as $t\to 0^+$. This assumption is standard in the analysis of time-fractional PDEs and guarantees the validity of termwise fractional differentiation and the series expansions employed in the GRPSM (see, e.g., Podlubny [10] and Kilbas et al. [11] for details).

Lemma 1 (Basic identities)**.** *For $n-1<\alpha\leq n$, $\gamma>-1$ and $t\geq 0$, the following hold:*

1. $D_t^{\alpha}c=0$*, for any constant* $c\in\mathbb{R}$*;*
2. $D_t^{\alpha}t^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}$*;*

3. $D_t^{m\alpha} t^{k\alpha} = \begin{cases} \dfrac{\Gamma(k\alpha+1)}{\Gamma((k-m)\alpha+1)} t^{(k-m)\alpha}, & k \geq m, \\ 0, & k < m, \end{cases}$
   where $D_t^{m\alpha}$ denotes m successive Caputo derivatives.

These fractional identities will later be used to evaluate the series coefficients and residual terms in the GRPSM expansion.

2.2. General Integral Transform Preliminaries

Definition 2 (General Laplace-like transform)**.** *Let $f(t)$ be an integrable function satisfying $|f(t)| \leq Me^{\delta t}$ for some constants $M, \delta > 0$. The* general integral transform *of $f(t)$, denoted $T\{f(t)\}$, is defined by*

$$T\{f(t)\}(s) = \nu(s) \int_0^\infty f(t)\, e^{-\omega(s)t}\, dt, \qquad \omega(s) > \delta,\ \nu(s) \neq 0, \tag{2}$$

where $\omega(s)$ and $\nu(s)$ are positive real functions. This definition includes the Laplace, Elzaki, Aboodh, Kamal, and several other transforms as special cases, depending on the choice of $\omega(s)$ and $\nu(s)$.

Lemma 2 (Properties of the general transform [8,12])**.** *Let f and g be piecewise continuous functions on $[0, \infty)$. If $F(s) = T\{f(t)\}$ and $G(s) = T\{g(t)\}$, then the following properties hold for all admissible $\omega(s)$ and $\nu(s)$:*

1. ***Linearity:*** $T\{c_1 f + c_2 g\} = c_1 F + c_2 G, c_1$ *and* c_2 *are arbitrary constants.*
2. ***Transform of a power:*** $T\{t^\alpha f(t)\} = \dfrac{\nu(s)\Gamma(1+\alpha)}{\omega(s)^{1+\alpha}} F(s)$, *for* $\alpha > 0$.
3. ***Initial-value limit:*** $\lim\limits_{\omega(s)\to\infty} \dfrac{\omega(s)}{\nu(s)} F(s) = f(0)$.
4. ***Transform of the Caputo derivative:*** *For* $0 < \alpha < 1, n \geq 1$

$$T\{D_t^{n\alpha} f(x,t)\} = \omega(s)^{n\alpha} T\{f(x,t)\} - \nu(s) \sum_{k=0}^{n-1} \omega(s)^{(n-k)\alpha-1} D_t^{k\alpha} f(x,0),$$

where $D_t^{n\alpha} = \underbrace{D_t^\alpha D_t^\alpha \dots D_t^\alpha}_{n \text{ times}}$.

These relations establish the bridge between the time-domain fractional operators and their transform-domain representations, which is fundamental to constructing the GRPSM framework in the next section.

3. Methodology: General Residual Power Series Method (GRPSM)

The Residual Power Series Method (RPSM) provides a systematic analytic framework for fractional partial differential equations (FPDEs) of the form

$$\mathcal{D}_t^\alpha \psi(x,t) = N[\psi(x,t)] + h(x,t), \qquad 0 < \alpha \leq 1, \tag{3}$$

subject to initial condition: $\psi(x,0) = f_0(x)$, where $\mathcal{D}_t^\alpha$ denotes the Caputo fractional derivative and $N[\psi(x,t)]$ is a nonlinear term of unknown function ψ and its derivatives with respect to x. $h(x,t)$ is a non-homogeneous function.

3.1. Fractional Power Series Representation

To encompass all Laplace-like transforms, we introduce the general integral transform

$$T\{f(t)\}(s) = \nu(s) \int_0^\infty e^{-\omega(s)t} f(t)\, dt. \tag{4}$$

With appropriate choices of these functions, (4) reproduces the Laplace, Elzaki, Aboodh, Sumudu, Kamal, Pourreza, and other related transforms.

Assume that $\psi(x,t)$ admits a fractional power series in t^α,

$$\psi(x,t) = \sum_{k=0}^{\infty} \frac{\phi_k(x)}{\Gamma(k\alpha+1)}\, t^{k\alpha}, \tag{5}$$

where $\phi_k(x)$ is the coefficients and $\phi_0(x) = \psi(x,0)$.

The corresponding transform in time domain $\Psi(x,s) = T\{\psi(x,t)\}$ of (5) is

$$\Psi(x,s) = \sum_{k=0}^{\infty} \frac{\nu(s)}{\omega(s)^{k\alpha+1}}\, \phi_k(x), \tag{6}$$

for $0 < \delta < \omega(s)$ when $\psi(x,t)$ is of exponential order.

Applying T to (3) and using its mapping property for Caputo derivatives in Lemma 2 gives

$$\omega(s)^\alpha \Psi(x,s) - \nu(s)\omega(s)^{\alpha-1}\psi(x,0) = T\{N[\psi(x,t)] + h(x,t)\}, \quad \omega(s) > \delta.$$

Therefore this results provide solution in the transformed domain

$$\Psi(x,s) = \frac{\nu(s)}{\omega(s)}\psi(x,0) + \frac{1}{\omega(s)^\alpha} T\{N[\psi(x,t)] + h(x,t)\}. \tag{7}$$

Substituting (5) into (3) produces the *residual function in time domain*

$$R(x,t) = \mathcal{D}_t^\alpha \psi(x,t) - N[\psi(x,t)] - h(x,t).$$

The coefficients are determined by enforcing

$$\mathcal{D}_t^{(k-1)\alpha} R(x,t)\big|_{t=0} = 0, \tag{8}$$

which yields a hierarchy of algebraic relations for $\{\phi_k\}$.

In order to find the coefficients ϕ_k, we introduce the *General residual function in the transformed domain* which quantifies the error between the truncated series $\Psi_m(x,s)$ and the transform solution (7) as follow

$$GR_m(x,s) = \sum_{k=1}^{m} \frac{\nu(s)}{\omega(s)^{k\alpha+1}} \phi_k(x) - \frac{1}{\omega(s)^\alpha} T\Big\{N[\psi(x,t)] + h(x,t)\Big\}. \tag{9}$$

3.2. Derivation of the Universal Coefficient Formula

The derivation proceeds by acting $\mathcal{D}_t^\alpha$ on the series (5) and expanding the right-hand side of (3) in powers of t^α. Term-by-term comparison of coefficients of equal order $t^{m\alpha}$ leads to a recursive formula that depends only on the fractional derivative of $N[\psi] + h$ evaluated at $t = 0$. For completeness, the argument is formalized below as Lemma 1, accompanied by its proof using induction on k.

Lemma 3 (Universal coefficient formula)**.** *Let $\psi(x,t)$ satisfy (3) and admit the expansion*

$$\psi(x,t) = \sum_{k=0}^{\infty} \frac{\phi_k(x)}{\Gamma(k\alpha+1)}\, t^{k\alpha},$$

in a neighborhood of $t = 0$. Assume that for some integer $M \geq 1$. The fractional derivatives $\mathcal{D}_t^{m\alpha}\psi(x,t)$ and $\mathcal{D}_t^{m\alpha}\big(N[\psi]+h\big)$ exist and are continuous at $t=0$ for all $m \leq M$. Then, for each integer $k \geq 1$

$$\phi_k(x) = \mathcal{D}_t^{(k-1)\alpha}\big(N[\psi(x,t)] + h(x,t)\big)\Big|_{t=0}. \tag{10}$$

Proof. The proof proceeds by induction on k. Setting $t = 0$ in (3) yields

$$\mathcal{D}_t^{\alpha}\psi(x,t)\big|_{t=0} = N[\psi(x,0)] + h(x,0),$$

and since $\mathcal{D}_t^{\alpha}\psi(x,t)\big|_{t=0} = \phi_1(x)$ from the series identity, we obtain (10) for $k = 1$.

Assume (10) holds for all indices up to $k = r$ with $r + 1 \leq M$. We show it holds for $k = r + 1$. Apply the operator $\mathcal{D}_t^{r\alpha}$ to both sides of Equation (3) and then apply to the result

$$\mathcal{D}_t^{(r+1)\alpha}\psi(x,t) = \mathcal{D}_t^{r\alpha}\big(N[\psi(x,t)] + h(x,t)\big).$$

Now substitute the series expansion for ψ into the left-hand side

$$\mathcal{D}_t^{(r+1)\alpha}\psi(x,t) = \sum_{j=0}^{\infty} \frac{\phi_{j+r+1}(x)}{\Gamma(j\alpha+1)}\, t^{j\alpha}.$$

Evaluating this identity at $t = 0$ annihilates all terms with $j \geq 1$, leaving only the constant term on the left.

$$\mathcal{D}_t^{(r+1)\alpha}\psi(x,t)\big|_{t=0} = \phi_{r+1}(x).$$

Thus

$$\phi_{r+1}(x) = \mathcal{D}_t^{r\alpha}\big(N[\psi(x,t)] + h(x,t)\big)\big|_{t=0},$$

which is exactly (10) for $k = r + 1$. This closes the induction. □

Before deriving the universal coefficient relations, we establish the convergence of the fractional power series obtained from the general transform (4). Let the *remainder function in the transformed domain* be defined as

$$R_n(x,s) = \Psi(x,s) - \sum_{k=0}^{n} \frac{\nu(s)}{\omega(s)^{k\alpha+1}}\, \phi_k(x), \tag{11}$$

where $\Psi(x,s) = T\{\psi(x,t)\}$ is the transform of $\psi(x,t)$.

We note that the notion of the general residual function (9) is closely related to the definition of the remainder function.

Theorem 1 (Boundedness of the remainder)**.** *Assume $\psi(x,t)$ is continuous on $I \times [0,\infty)$ and of exponential order, i.e., $|\psi(x,t)| \leq Me^{\delta t}$ for some $\delta, M > 0$. If $T\{\mathcal{D}_t^{m\alpha}\psi(x,t)\}$ exists for $m \leq n$ and there exists $K(x) > 0$ such that*

$$\Big|\omega(s)T\{\mathcal{D}_t^{n\alpha}\psi(x,t)\}\Big| \leq K(x), \qquad x \in I,\ \omega(s) \in (\delta,\beta), \tag{12}$$

then the remainder $R_n(x,s)$ in (11) satisfies the inequality

$$|R_n(x,s)| \leq \frac{K(x)}{\omega(s)^{n\alpha+1}}, \qquad x \in I,\ \omega(s) \in (\delta,\beta). \tag{13}$$

Proof. Based on Lemma 3, the coefficient $\phi_n(x)$ of the series in (6) becomes $\phi_n(x) = D_t^{n\alpha}\psi(x,0)$. Therefore, the remainder $R_n(x,s)$ can be written by

$$\begin{aligned} R_n(x,s) &= \Psi(x,s) - \sum_{k=0}^{n} \frac{\nu(s)}{\omega(s)^{k\alpha+1}} D_t^{k\alpha}\psi(x,0) \\ \omega(s)^{n\alpha+1}R_n(x,s) &= \omega(s)\left(\omega(s)^{n\alpha}\Psi(x,s) - \nu(s)\sum_{k=0}^{n}\omega^{(n-k)\alpha-1}D_t^{k\alpha}\psi(x,0)\right) \\ &= \omega(s)T\{D_t^{n\alpha}\psi(x,t)\}. \end{aligned}$$

The last equation is obtained by part 4 of Lemma 2. It follows from (12) that

$$|R_n(x,s)| < \frac{K(x)}{\omega(s)^{n\alpha+1}}, \quad x \in I,\ \delta < \omega(s) < \beta.$$

□

Remark 2. *The constant $K(x)$ in Theorem 13 reflects the local growth of the higher-order fractional derivative $D_t^{(n+1)\alpha}\psi(x,t)$ and therefore governs the convergence rate of the residual series. For most linear and weakly nonlinear time-fractional PDEs with sufficiently smooth solutions, this derivative remains bounded within the finite domain of interest. Consequently, the remainder term satisfies*

$$|R_n(x,s)| \le K(x)\frac{|\omega|^{(n+1)\alpha+1}}{(n+1)!},$$

which ensures that $R_n(x,s) \to 0$ rapidly as n increases.

For instance, for the time-fractional diffusion equation Ref. [10]

$$D_t^{\alpha}\psi(x,t) = \psi_{xx}(x,t),$$

the fractional derivatives behave approximately as

$$D_t^{(n+1)\alpha}\psi(x,t) \sim C\,t^{1-(n+1)\alpha}.$$

Hence, for any finite t, this term remains bounded and decays rapidly with n, ensuring that $K(x)$ is finite and the residual power series converges uniformly. This interpretation provides an intuitive understanding of how the analytical bound guarantees convergence without the need for additional numerical verification.

Remark 3. *Inequality (13) ensures that for $n \ge 1$*

$$\lim_{\omega(s)\to\infty} \omega(s)^{n\alpha+1}R_n(x,s) = 0, \tag{14}$$

hence the transform series (6) and the time-domain series (5) are convergent for $\omega(s) > \delta$. We note that (14) is equivalent to

$$\lim_{\omega(s)\to\infty} \omega(s)^{n\alpha+1}GR_n(x,s) = 0, \quad n \ge 1. \tag{15}$$

Proposition 1 (Universal Coefficient Formula for the GRPSM)**.** *Let $\psi(x,t)$ be a series solution of the FPDE (3) and satisfy the hypotheses of Theorem 1 and Lemma 3. Then the coefficients of the solution are*

$$\phi_k(x) = \mathcal{D}_t^{(k-1)\alpha}\left(N[\psi(x,t)] + h(x,t)\right)\Big|_{t=0}, \quad k \ge 1. \tag{16}$$

Proof. The proof proceeds by using the general residual function in the transform-domain (9). For $k = 1$, multiply $GR_1(x,s)$ by $\frac{\omega^{\alpha+1}}{\nu(s)}$ and take limit as $\omega(s) \to \infty$

$$\lim_{\omega(s)\to\infty} \frac{\omega^{\alpha+1}}{\nu(s)} GR_1(x,s) = \phi_1(x) - \lim_{\omega(s)\to\infty} \frac{\omega(s)}{\nu(s)} T\Big\{N[\psi(a,t)] + h(x,t)\Big\}.$$

From (15), the left-hand limit vanishes, whereas the right-hand limit is obtained via part 3 of Lemma 2. Therefore,

$$\phi_1(x) = \Big(N[\psi(a,t)] + h(x,t)\Big)\Big|_{t=0}. \tag{17}$$

Assume that (16) holds for $k \le m-1$,

$$\phi_k(x) = D_t^{(k-1)\alpha}\Big(N[\psi(a,t)] + h(x,t)\Big)\Big|_{t=0}. \tag{18}$$

Analogously, multiply $GR_m(x,s)$ by $\frac{\omega^{m\alpha+1}}{\nu(s)}$ and take limit as $\omega(s) \to \infty$

$$\lim_{\omega(s)\to\infty} \frac{\omega^{m\alpha+1}}{\nu(s)} GR_m(x,s) = \phi_m(x) - \lim_{\omega(s)\to\infty} \frac{\omega(s)}{\nu(s)} \Bigg[\omega(s)^{(m-1)\alpha} T\Big\{N[\psi(a,t)] + h(x,t)\Big\} - \nu(s)\sum_{k=1}^{m-1} \omega(s)^{(m-k)\alpha-1}\phi_k(x)\Bigg]. \tag{19}$$

According to (15), the left-hand limit vanishes, while the right-hand limit results from substituting (18) into (19) and using part 4 Lemma 2.

$$\begin{aligned}
\phi_m(x) &= \lim_{\omega(s)\to\infty} \frac{\omega(s)}{\nu(s)} \Bigg[\omega(s)^{(m-1)\alpha} T\Big\{N[\psi(a,t)] + h(x,t)\Big\} \\
&\qquad - \nu(s)\sum_{k=1}^{m-1} \omega(s)^{(m-k)\alpha-1} D_t^{(k-1)\alpha}\Big(N[\psi(a,t)] + h(x,t)\Big)\Big|_{t=0}\Bigg] \\
&= \lim_{\omega(s)\to\infty} \frac{\omega(s)}{\nu(s)} \Bigg[\omega(s)^{(m-1)\alpha} T\Big\{N[\psi(a,t)] + h(x,t)\Big\} \\
&\qquad - \nu(s)\sum_{k=0}^{m-2} \omega(s)^{(m-1-k)\alpha-1} D_t^{(k)\alpha}\Big(N[\psi(a,t)] + h(x,t)\Big)\Big|_{t=0}\Bigg] \\
&= \lim_{\omega(s)\to\infty} \frac{\omega(s)}{\nu(s)} T\{D_t^{(m-1)\alpha}\Big(N[\psi(a,t)] + h(x,t)\Big)\} \\
&= D_t^{(m-1)\alpha}\Big(N[\psi(a,t)] + h(x,t)\Big)\Big|_{t=0}.
\end{aligned}$$

The final equality follows directly from part 3 of Lemma 2, thereby completing the proof. □

Remark 4. *Suppose the term N on the right-hand side of Equation (3) is linear, written as $N[u] = L[u]$, where L denotes a linear operator. In this case, the formula for the coefficients in Equation (16) becomes*

$$\phi_k(x) = D_t^{(k-1)\alpha}\left(\sum_{m=0}^{k-1} L\left[\frac{\phi_m(x)t^{m\alpha}}{\Gamma(1+m\alpha)}\right] + h(x,t)\right)\Bigg|_{t=0}, \quad k = 1,2,3,\ldots.$$

Using Lemma 1, we derive the coefficient formulas for a time-fractional linear PDE as follows:

$$\phi_k(x) = L[\phi_{k-1}(x)] + D_t^{(k-1)\alpha} h(x,t)\Big|_{t=0}, \quad k = 1, 2, 3, \ldots. \tag{20}$$

3.3. Discussion and Implications

The preceding analysis shows that the residual-based and transform-based formulations of the GRPSM are analytically equivalent representations of the same recursive mechanism. In the residual formulation, the coefficients $\phi_k(x)$ are obtained from the hierarchy $D_t^{(k-1)\alpha} R(x,t)\big|_{t=0} = 0$, whereas the transform-based approach—employed in Laplace, Elzaki, and Aboodh versions of the RPSM— derives them by matching powers of $\omega(s)^{-1}$. Although the derivations differ in appearance, both lead to identical recurrence relations, establishing the analytical *transform invariance* of the GRPSM. This invariance refers to a structural equivalence between formulations rather than a numerical property: it follows directly from the identical recursive structure revealed in Proposition 1.

The analytical correspondence between the two formulations is summarized in Table 1. The table highlights that each formulation generates the same coefficient sequence $\{\phi_k(x)\}_{k=0}^{\infty}$, differing only in the domain—time versus transform—in which the residual constraints are expressed.

Table 1. Analytical equivalence between residual- and transform-based formulations of the GRPSM.

Aspect	Residual-Based Formulation	Transform-Based Formulation
Governing relation	Residual hierarchy $D_t^{(k-1)\alpha} R(x,t)\big|_{t=0} = 0$ defines the coefficients $\phi_k(x)$.	Coefficients $\phi_k(x)$ obtained by matching powers of $\omega(s)^{-1}$ in the transformed residual function.
Coefficient recursion	Generated directly from the local residual conditions in the time domain.	Derived from algebraic identities of the Laplace-like kernel pair (ω, ν).
Transform kernel dependence	No transform required; formulation entirely in the physical domain.	Depends on the kernel asymptotics of the chosen transform, but produces the same $\{\phi_k(x)\}$ sequence.
Interpretation	Local differential constraint on the residual function.	Equivalent algebraic representation under any admissible Laplace-like transform.
Resulting series solution	$\psi(x,t) = \sum_{k=0}^{\infty} \phi_k(x) t^{k\alpha}$	Identical truncated series obtained via inverse transform.

It is also worth emphasizing that the analytical equivalence established here implies a numerical independence among Laplace-like transform variants. Because the coefficient recursions are identical in closed form, each transform-based implementation yields the same truncated series when evaluated numerically. Hence, additional numerical verification is unnecessary—the invariance is inherent to the analytical structure of the method rather than to computational precision.

The proposed GRPSM should therefore be viewed as a conceptual generalization rather than a replacement for existing implementations. Its contribution lies in revealing that all Laplace-like RPSM variants are analytically equivalent under a unified coefficient recursion. In this sense, the "computational simplicity" achieved by the GRPSM refers to a reduction in symbolic manipulations and the elimination of redundant limit evaluations across different transforms, rather than any improvement in numerical runtime. This unified perspective formally justifies the use of the classical RPSM in practice and provides a common analytical reference for the design of transform-based algorithms. The next section illustrates these theoretical results through representative examples.

4. Illustrative Examples

To demonstrate the practicality and efficiency of proposed GRPSM, this section presents three representative examples restricted to time-fractional PDEs of order $0 < \alpha \leq 1$. Each example verifies the transform-invariance property and highlights how the explicit coefficient formulas (16) simplify the computation of approximate analytic solutions.

Example 1 considers a linear time-fractional Black–Scholes equation, illustrating the basic procedure and comparison with earlier Laplace- and Aboodh-RPSM results [13,14]. Example 2 addresses the time-fractional biological population diffusion equation. We unequivocally demonstrate that the proposed formula yields a robust solution consistent with the findings reported in the literature that utilized the Elzaki Transform residual power series method (Elzaki-RPSM) (Ref. [3]). Example 3 deals with a nonlinear time-fractional Burgers–Fisher equation, showing that the same coefficient recursion effectively handles nonlinearities without modifying the transform framework.

Example 1. *Consider a time-fractional Black–Scholes equation of order $0 < \alpha \leq 1$ [13,14]:*

$$\begin{aligned} D_t^\alpha V(y,t) &= -0.08(2\sin y)^2 y^2\, V_{yy} - 0.06 y V_y + 0.06 V, \\ V(y,0) &= \max\{y - 25e^{-0.06}, 0\}. \end{aligned} \tag{21}$$

The expression on the RHS of this equation is linear:

$$L[V] = -0.08(2\sin y)^2 y^2\, V_{yy} - 0.06 y V_y + 0.06 V.$$

Based on the coefficient Formula (20), the following coefficients are obtained:

$$\phi_1(y) = L[V(y,0)] = L[\max\{y - 25e^{-0.06}, 0\}].$$

Therefore,

$$\phi_1(y) = 0.06\big(\max\{y - 25e^{-0.06}, 0\} - y\big).$$

Similarly, to determine the coefficient $\phi_2(y)$ in Equation (20),

$$\begin{aligned} \phi_2(y) &= L[\phi_1(y)] \\ &= -0.08(2\sin y)^2 y^2 \phi_1''(y) - 0.06 y\, \phi_1'(y) + 0.06\phi_1(y) \\ &= 0.06^2\big(\max\{y - 25e^{-0.06}, 0\} - y\big). \end{aligned}$$

Likewise, the remaining coefficients $\phi_k(y)$ for $k = 3, 4, \ldots$ can be determined by

$$\phi_k(y) = 0.06^k(\max\{y - 25e^{-0.06}, 0\} - y). \tag{22}$$

Finally, substitute these coefficients in Equation (5), the series solution for the time-fractional Black–Scholes Equation (21) is presented below:

$$V(y,t) = 0.06(\max\{y - 25e^{-0.06}, 0\} - y) + \sum_{n=1}^{\infty} 0.06^n(\max\{y - 25e^{-0.06}, 0\} - y)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \tag{23}$$

Our solution aligns with both [13] (using the traditional RPSM) and [14] (using the Aboodh-RPSM), but with a more efficient approach for deriving the coefficients.

Example 2. *Consider the time-fractional biological population diffusion equation [3]:*

$$D_t^\alpha u(x,y,t) = [u^2]_{xx} + [u^2]_{yy} + cu(1 - ru), \tag{24}$$

subject to the initial condition:

$$u(x,y,0) = \phi_0(x,y) = e^{\sqrt{\frac{cr}{8}}(x+y)}. \tag{25}$$

where $0 < \alpha \leq 1$ *and* c *is a given constant.*

The nonlinear RHS term is $N[u] = [u^2]_{xx} + [u^2]_{yy} + cu(1 - ru)$.

By using Equation (16), we obtain

$$\begin{aligned}\phi_1(x,y) &= N[\phi_0(x,y)] \\ &= [\phi_0^2]_{xx} + [\phi_0^2]_{yy} + c\phi_0(1 - r\phi_0)\end{aligned}$$

Based on the initial function $\phi_0(x,y)$ *in Equation (25), we obtain*

$$\phi_1(x,y) = c\phi_0 = ce^{\sqrt{\frac{cr}{8}}(x+y)}. \tag{26}$$

Utilizing the coefficient Formula (16) when $k = 2$, *we obtain the second coefficient*

$$\phi_2(x,y) = D_t^\alpha\left(N\left[\underbrace{\phi_0 + \frac{\phi_1 t^\alpha}{\Gamma(1+\alpha)}}_{=:\Theta_1} \right] \right)\Bigg|_{t=0} = D_t^\alpha\left([\Theta_1^2]_{xx} + [\Theta_1^2]_{yy} + c\Theta_1 - cr\Theta_1^2 \right)\Big|_{t=0}, \tag{27}$$

We now compute the following terms by using part (3) of Lemma 1:

$$D_t^\alpha(\Theta_1)\Big|_{t=0} = D_t^\alpha\left(\phi_0 + \frac{\phi_1 t^\alpha}{\Gamma(1+\alpha)} \right)\Bigg|_{t=0} = \phi_1 \tag{28}$$

and

$$D_t^\alpha(\Theta_1^2)\Big|_{t=0} = D_t^\alpha\left(\phi_0^2 + 2\frac{\phi_0\phi_1 t^\alpha}{\Gamma(1+\alpha)} + \frac{\phi_1^2 t^{2\alpha}}{(\Gamma(1+\alpha))^2} \right)\Bigg|_{t=0} = 2\phi_0\phi_1. \tag{29}$$

Placing Equations (28) and (29) into Equation (27), we obtain the second coefficients

$$\phi_2(x,y) = 2[\phi_0\phi_1]_{xx} + 2[\phi_0\phi_1]_{yy} + c\phi_1 - 2cr\phi_0\phi_1. \tag{30}$$

By substituting ϕ_0 *from Equation (25) and* ϕ_1 *from Equation (26) into Equation (30), we obtain the second coefficient as shown below:*

$$\phi_2(x,y) = c^2 e^{\sqrt{\frac{cr}{8}}(x+y)}. \tag{31}$$

To determine subsequent coefficients $\phi_k(x,y),\ k = 3,4,\ldots$, *we use the coefficients Formula (16)*

$$\begin{aligned}\phi_k(x,y) &= D_t^{(k-1)\alpha}\left(N\left[\underbrace{\sum_{n=0}^{k-1} \frac{\phi_n(x,y) t^{n\alpha}}{\Gamma(1+n\alpha)}}_{=:\Theta_{k-1}} \right] \right)\Bigg|_{t=0} \\ &= D_t^{(k-1)\alpha}\left([\Theta_{k-1}^2]_{xx} + [\Theta_{k-1}^2]_{yy} + c\Theta_{k-1} - cr\Theta_{k-1}^2 \right)\Big|_{t=0}.\end{aligned} \tag{32}$$

Using Lemma 1, we compute the following terms

$$D_t^{(k-1)\alpha}(\Theta_{k-1})\Big|_{t=0} = D_t^{(k-1)\alpha}\left(\sum_{n=0}^{k-1}\frac{\phi_n(x,y)t^{n\alpha}}{\Gamma(1+n\alpha)}\right)\Bigg|_{t=0} = \phi_{k-1}(x,y) \tag{33}$$

and

$$\begin{aligned} D_t^{(k-1)\alpha}(\Theta_{k-1}^2)\Big|_{t=0} &= D_t^{(k-1)\alpha}\left(\sum_{m=0}^{k-1}\sum_{n=0}^{k-1}\frac{t^{(m+n)\alpha}}{\Gamma(1+m\alpha)\Gamma(1+n\alpha)}\phi_m(x,y)\phi_n(x,y)\right)\Bigg|_{t=0} \\ &= \sum_{m=0}^{k-1}\frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+m\alpha)\Gamma(1+(k-1-m)\alpha))}\phi_m(x,y)\phi_{k-1-m}(x,y). \end{aligned} \tag{34}$$

Placing Equations (33) and (34) into Equation (32), to obtain

$$\begin{aligned} \phi_k(x,y) = \sum_{m=0}^{k-1}\frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+m\alpha)\Gamma(1+(k-1-m)\alpha))}\Big([\phi_m\phi_{k-1-m}]_{xx} + [\phi_m\phi_{k-1-m}]_{yy} \\ - cr\phi_m\phi_{k-1-m}(x,y)\Big) + c\phi_{k-1}, \quad k = 3,4,\ldots. \end{aligned} \tag{35}$$

It is easy to see that if $\phi_n(x) = c^n e^{\sqrt{\frac{cr}{8}}(x+y)}, \quad n = 0,1,\ldots,k-1$, *then*

$$[\phi_m\phi_{k-1-m}]_{xx} + [\phi_m\phi_{k-1-m}]_{yy} - cr\phi_m\phi_{k-1-m} = 0,$$

for all $k = 3,4,\ldots$ *and* $m = 0,1,\ldots,k-1$.

Hence, the $k-$*th coefficient in Equation (35) becomes*

$$\phi_k(x,y) = c\phi_{k-1}(x,y), \quad k = 3,4,\ldots.$$

Based on the coefficient ψ_2 *in Equation (31) , we can conclude that the coefficients of the series solution of (24) are*

$$\phi_n(x,y) = c^n e^{\sqrt{\frac{cr}{8}}(x+y)}, \quad n = 0,1,2,\ldots.$$

The coefficients obtained in this work are identical to those presented in [15] for the Elzaki-RPSM. However, our derivation offers a more streamlined and robust approach for determining coefficients within the context of series solutions. In particular, the $k-$*th approximate solution is*

$$u_n(x,y,t) = e^{\sqrt{\frac{cr}{8}}(x+y)}\sum_{m=0}^{k}\frac{(ct^\alpha)^m}{\Gamma(1+m\alpha)}.$$

As $n \to \infty$, *the exact solution of the time-fractional biological population diffusion Equation (24) is*

$$u(x,y,t) = e^{\sqrt{\frac{cr}{8}}(x+y)}E_{\alpha,1}(ct^\alpha),$$

where $E_{\alpha,\beta}(t)$ *is the Mittag-Leffler function defined as* $E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty}\frac{t^m}{\Gamma(\beta+m\alpha)}$.

Example 3. *Consider the non-homogeneous nonlinear time-fractional gas dynamics equation [8]:*

$$D_t^\alpha u(x,t) = cu(1-u) - uu_x + h(x,t), \quad 0 < \alpha \leq 1, \tag{36}$$

subject to the initial condition: $u(x,0) = \phi_0(x)$, *where* c *is a given constant and* $h(x,t)$ *is a continuous function.*

Note that $h(x,t)$ is the non-homogeneous term and the nonlinear term is

$$N[u] = cu(1-u) - uu_x. \tag{37}$$

To determine the coefficients, we use Formula (16) to obtain ϕ_1:

$$\phi_1(x) = N[\phi_0(x)] + h(x,0) = c\phi_0(x)(1-\phi_0(x)) - \phi_0(x)\phi_0'(x) + h(x,0). \tag{38}$$

For the subsequent coefficients, we use Formula (16):

$$\begin{aligned}\phi_2(x,t) &= D_t^\alpha \left(N\left[\underbrace{\phi_0(x) + \frac{\phi_1(x)t^\alpha}{\Gamma(1+\alpha)}}_{=:\Theta_1} \right] \right) \Bigg|_{t=0} \\ &= D_t^\alpha (c\Theta_1(1-\Theta_1) - \Theta_1[\Theta_1]_x)\Big|_{t=0} + D_t^\alpha h(x,t)\Big|_{t=0}. \end{aligned} \tag{39}$$

Observe the following terms, by using part (3) of Lemma 1, we obtain

$$\begin{aligned} D_t^\alpha(c\Theta_1(1-\Theta_1))\Big|_{t=0} &= cD_t^\alpha(\Theta_1)\Big|_{t=0} - cD_t^\alpha\left(\Theta_1^2\right)\Big|_{t=0} \\ &= cD_t^\alpha\left(\phi_0(x) + \frac{\phi_1(x)t^\alpha}{\Gamma(1+\alpha)}\right)\Bigg|_{t=0} \\ &\quad - cD_t^\alpha\left(\phi_0^2 + 2\frac{\phi_0\phi_1 t^\alpha}{\Gamma(1+\alpha)} + \frac{\phi_1^2 t^{2\alpha}}{(\Gamma(1+\alpha))^2}\right)\Bigg|_{t=0} \\ &= c\phi_1(x) - 2c\phi_0(x)\phi_1(x), \end{aligned} \tag{40}$$

and

$$\begin{aligned} D_t^\alpha(\Theta_1[\Theta_1]_x)\Big|_{t=0} &= D_t^\alpha\left(\phi_0(x)\phi_0'(x) + \frac{\phi_0(x)\phi_1'(x)\,t^\alpha}{\Gamma(1+\alpha)} + \frac{\phi_0'(x)\phi_1(x)\,t^\alpha}{\Gamma(1+\alpha)} + \frac{\phi_1'(x)\phi_1(x)t^{2\alpha}}{(\Gamma(1+\alpha))^2}\right)\Bigg|_{t=0} \\ &= \phi_0(x)\phi_1'(x) + \phi_0'(x)\phi_1(x). \end{aligned} \tag{41}$$

Substituting Equations (40) and (41) into Equation (39), we obtain the second coefficient

$$\phi_2(x) = c\phi_1(x) - 2c\phi_0(x)\phi_1(x) - \phi_0(x)\phi_1'(x) - \phi_0'(x)\phi_1(x) + D_t^\alpha h(x,t)\Big|_{t=0}, \tag{42}$$

which is the same as the coefficient obtained in [8].

Likewise, to determine the third coefficient, we use Formula (16):

$$\begin{aligned}\phi_3(x) &= D_t^{2\alpha} \left(N\left[\underbrace{\phi_0(x) + \frac{\phi_1(x)t^\alpha}{\Gamma(1+\alpha)} + \frac{\phi_2(x)t^{2\alpha}}{\Gamma(1+2\alpha)}}_{=:\Theta_2} \right] \right) \Bigg|_{t=0} \\ &= D_t^{2\alpha} (c\Theta_2(1-\Theta_2) - \Theta_2[\Theta_2]_x)\Big|_{t=0} + D_t^{2\alpha} h(x,t)\Big|_{t=0}. \end{aligned} \tag{43}$$

Evaluate the following terms, we use part (3) of Lemma 1 to calculate fractional derivatives of the fractional power functions:

$$D_t^{2\alpha}(\Theta_2)|_{t=0} = \phi_2(x), \tag{44}$$

$$\begin{aligned} D_t^{2\alpha}(\Theta_2^2)|_{t=0} &= D_t^{2\alpha}\left(\sum_{n=0}^{2}\frac{\phi_n(x)t^{n\alpha}}{\Gamma(1+n\alpha)}\right)^2 \\ &= D_t^{2\alpha}\left(\sum_{m=0}^{2}\sum_{n=0}^{2}\frac{t^{(m+n)\alpha}}{\Gamma(1+m\alpha)\Gamma(1+n\alpha)}\phi_m\phi_n\right)\Bigg|_{t=0} \\ &= \sum_{m=0}^{2}\frac{\Gamma(1+2\alpha)}{\Gamma(1+m\alpha)\Gamma(1+(2-m)\alpha))}\phi_m\phi_{2-m} \\ &= 2\phi_0\phi_2 + \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\phi_1^2, \end{aligned} \tag{45}$$

and

$$\begin{aligned} D_t^{2\alpha}\left(\Theta_2[\Theta_2]_x\right)\Big|_{t=0} &= D_t^{2\alpha}\left(\sum_{m=0}^{2}\sum_{n=0}^{2}\frac{t^{(m+n)\alpha}}{\Gamma(1+m\alpha)\Gamma(1+n\alpha)}\phi_m\phi_n'\right)\Bigg|_{t=0} \\ &= \sum_{m=0}^{2}\frac{\Gamma(1+2\alpha)}{\Gamma(1+m\alpha)\Gamma(1+(2-m)\alpha))}\phi_m\phi_{2-m}' \\ &= \phi_0\phi_2' + \phi_0'\phi_2 + \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\phi_1\phi_1'. \end{aligned} \tag{46}$$

Substituting Equations (44)–(46) into Equation (43), the third coefficient becomes

$$\phi_3(x) = c\phi_2 - 2c\phi_0\phi_2 - c\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\phi_1^2 - \phi_0\phi_2' - \phi_0'\phi_2 - \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\phi_1\phi_1' + D_t^{2\alpha}h(x,t)\Big|_{t=0}. \tag{47}$$

which is different to the third coefficient that appeared in [8].

For the remaining coefficients $\phi_k(x), k = 4, 5, \ldots$, we use the coefficients Formula (16) to evaluate them:

$$\begin{aligned} \phi_k(x) &= D_t^{(k-1)\alpha}\left(N\left[\underbrace{\sum_{n=0}^{k-1}\frac{\phi_n(x)t^{n\alpha}}{\Gamma(1+n\alpha)}}_{=:\Theta_{k-1}}\right] + h(x,t)\right)\Bigg|_{t=0} \\ &= D_t^{(k-1)\alpha}(c\Theta_{k-1}(1-\Theta_{k-1}) - \Theta_{k-1}[\Theta_{k-1}]_x)\Big|_{t=0} + D_t^{(k-1)\alpha}h(x,t)\Big|_{t=0}. \end{aligned} \tag{48}$$

To compute the following terms, we refer to Equations (33) and (34) in Example 2

$$\begin{aligned} &D_t^{(k-1)\alpha}\left(c\Theta_{k-1}(1-\Theta_{k-1})\right)\Big|_{t=0} \\ &\quad = c\phi_{k-1}(x) - c\sum_{m=0}^{k-1}\frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+m\alpha)\Gamma(1+(k-1-m)\alpha))}\phi_m\phi_{k-1-m} \end{aligned} \tag{49}$$

and

$$D_t^{(k-1)\alpha}(\Theta_{k-1}[\Theta_{k-1}]_x)\Big|_{t=0} = \sum_{m=0}^{k-1}\frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+m\alpha)\Gamma(1+(k-1-m)\alpha))}\phi_m\phi_{k-1-m}'. \tag{50}$$

Substituting Equations (49) and (50) into Equation (48) yields

$$\phi_k = c\phi_{k-1} - \sum_{m=0}^{k-1} \frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+m\alpha)\Gamma(1+(k-1-m)\alpha))}\left(c\phi_m\phi_{k-1-m} + \phi_m\phi'_{k-1-m}\right) + D_t^{(k-1)\alpha}h(x,t)\Big|_{t=0}, \tag{51}$$

for $k = 4, 5, \ldots$. In the following, we present the coefficients derived from Formula (51):

$$\begin{aligned}
\phi_4 &= c\phi_3 - 2c\phi_0\phi_3 - (2c\phi_1\phi_2 + \phi_1\phi'_2 + \phi'_1\phi_2)\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - (\phi_0\phi'_3 + \phi'_0\phi_3) \\
&\quad + D_t^{3\alpha}h(x,t)\Big|_{t=0}, \\
\phi_5 &= c\phi_4 - 2c\phi_0\phi_4 - (2c\phi_1\phi_3 + \phi_1\phi'_3 + \phi'_1\phi_3)\frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} \\
&\quad - (c\phi_2^2 + \phi_2\phi'_2)\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2} - (\phi_0\phi'_4 + \phi'_0\phi_4) + D_t^{4\alpha}h(x,t)\Big|_{t=0}, \\
\phi_6 &= c\phi_5 - 2c\phi_0\phi_5 - (2c\phi_1\phi_4 + \phi_1\phi'_4 + \phi'_1\phi_4)\frac{\Gamma(1+5\alpha)}{\Gamma(1+\alpha)\Gamma(1+4\alpha)} \\
&\quad - (2c\phi_2\phi_3 + \phi_2\phi'_3 + \phi'_2\phi_3)\frac{\Gamma(1+5\alpha)}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)}(\phi_0\phi'_5 + \phi'_0\phi_5) + D_t^{5\alpha}h(x,t)\Big|_{t=0}.
\end{aligned}$$

In this example, the truncated series with only a few terms already reproduces the expected analytical behavior, indicating rapid convergence of the GRPSM formulation. This serves as a compact indicator of computational efficiency without the need for additional numerical comparison. Furthermore, we will utilize coefficients Formula (16) to find the solution for the gas dynamics Equation (36) with the following choices of c, $u(x,0)$ and $h(x,t)$:

- *$c = 1, u(x,0) = \phi_0 = e^{-x}$ and $h(x,t) = 0$, the coefficients are $\phi_i(x) = e^{-x}$, $i = 0, 1, 2, \ldots$, we have the final solution*

$$u(x,t) = e^{-x}\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}.$$

- *$c = \ln b, u(x,0) = \phi_0 = b^{-x}$ and $h(x,t) = 0$, the coefficients are $\phi_i(x) = b^{-x}(\ln b)^i$, $i = 0, 1, 2, \ldots$. The obtained series solution is*

$$u(x,t) = b^{-x}\sum_{n=0}^{\infty}\frac{(\ln b)^n\, t^{n\alpha}}{\Gamma(1+n\alpha)}.$$

- *$c = 1, u(x,0) = \phi_0 = 1 - e^{-x}$ and $h(x,t) = -e^{-x+t}$, the coefficients are $\phi_i(x) = e^{-x}$, $i = 0, 1, 2, \ldots$, we have the final solution*

$$u(x,t) = 1 - e^{-x}\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}.$$

5. Conclusions

This paper presents the General Residual Power Series Method (GRPSM) as a conceptual framework that unifies the conventional RPSM and its Laplace-like variants under a single analytical formulation. By deriving explicit coefficient formulas, the GRPSM eliminates redundant limit evaluations and clarifies the mathematical structure underlying the recursive construction of residual power series solutions. A central result, given by Proposition 1, demonstrates that all admissible Laplace-like transforms lead to identical coefficient recursions, thereby establishing the analytical transformation invariance of the GRPSM. This invariance is a structural property of the method rather than a numerical one and follows directly from the equivalence of the residual- and transform-based formulations.

The unification of RPSM-type methods under the GRPSM framework highlights that the essential mechanism of these approaches originates from the residual formulation itself. In this sense, the classical RPSM remains the simplest and most direct form of the method, while the GRPSM provides a formal justification for its transform-invariant behavior. The "computational simplicity" achieved by the GRPSM should be interpreted as a reduction in symbolic manipulations and analytical redundancy rather than an improvement in numerical runtime.

Overall, the GRPSM establishes a unified and transparent theoretical foundation for understanding transform-based extensions of the RPSM. This conceptual clarification provides a consistent reference for future analytical and algorithmic developments in solving time-fractional and related differential equations.

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