Ultrasound Image Enhancement by Means of a Variational Approach

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ABSTRACT

Ultrasound images provide the clinicians with non-invasive, low cost and real-time images which can help them in diagnosis, planning and therapy. However, the ultrasonic wave encounters rough interfaces which implies the scattering and leads to the noise which is called speckle noise. The speckle noise can be explained statistically by the Rayleigh distribution:

$$P(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}.$$

In order to reduce the speckle noise in ultrasound images, we use the calculus of variations (COV). In general, COV provides the technique to find an unknown function which optimizes the functional. To optimize the ultrasound image, we seek for a minimizer of the integral equation:

$$E(u) = \iint_\Omega \left(\sqrt{u_x^2 + u_y^2} + \frac{\tilde{u}^2}{u^2} + \ln u\right) dxdy,$$

where u is the desired image, $\tilde{u}$ is the observed image and Ω is the image domain. By COV, the above problem is equivalent to finding the solution of the equation:

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}}\right) + \left(\frac{2\tilde{u}^2}{u^3} - \frac{1}{u}\right) = 0,$$

which is an Euler-Lagrange equation. The method for finding the numerical solution is the gradient descent method. The results show that the noise in the ultrasound images is reduced.

Keywords : calculus of variations, Euler-Lagrange equation, gradient descent method, Rayleigh distribution, speckle noise, ultrasound image.

INTRODUCTION

Medical ultrasonography (sonography) is an ultrasound-based diagnostic imaging technique used to visualize muscles and internal organs, their size, structures and possible pathologies or lesions. Ultrasound images provide the clinicians with non-invasive, low cost and real-time images which can help them in diagnosis, planning and therapy. The creation of an image from ultrasound is done in three steps which are producing a sound wave, receiving echoes, and interpreting those echoes. However, the ultrasonic wave encounters rough interfaces which implies the scattering and leads to the noise which is called speckle noise. The speckle can be explained statistically by the Rayleigh distribution which is described in section 3. In order to improve quality of a noisy ultrasound image, the approach for denoising is based on calculus of variations (COV). Section 4 explains that COV provides the technique to find an unknown function which optimizes the functional. Furthermore, in section 4, there is the explanation about the ROF model which is a prototype variational model for image denoising. This research develops the ROF model for speckle noise reduction in the ultrasound image. The proposed model is described in section 5. The numerical method to approximation the solution of the proposed model is gradient descent method and the numerical solution is showed in section 6. Numerical results and conclusion are presented in section 7 and section 8, respectively.

OBJECTIVE

In the manuscript, the method for reducing speckle noise is studied. The developed method is applied to denoise two dimensional ultrasound videos. A prototype software for demonstration and validation is implemented also.

METHODOLOGY

1. SPECKLE NOISE

Speckle is a random pattern which has a negative impact on coherent imaging, including ultrasound imaging. It is a result of the superposition of many waves, having different phases. Speckle occurs in ultrasound image because the ultrasonic wave encounters rough surfaces that implies the scattering of waves, each scattered wave from a rough surface has a different phase which bring to the speckle forming. Goodman (1985) shows that we can use the Rayleigh distribution to describe the speckle statistically. The *Rayleigh distribution* is a continuous probability distribution. It can arise when a two-dimensional vector has elements that are normally distributed, uncorrelated and they both have zero mean and have equal variance. The vector's magnitude will then has a Rayleigh distribution. The Rayleigh probability density function is

$$P(r) = \frac{r}{\sigma^2} e^{-r^2/\left(2\sigma^2\right)},$$

where r is non - negative number and σ is parameter.

2. CALCULUS OF VARIATIONS

Calculus of variations is field of mathematics that deals with functionals, as opposed to ordinary calculus which deals with functions. Such functionals can be formed as integrals involving an unknown function and its derivatives.The interest is in extremal functions making the functional attain an extremum value. The candidates in the competition for an extremum are functions. For example the problem involves finding the extremum of integrals of the form

$$I = \iint_{\Omega} F(x, y, u, u_x, u_y)\, dxdy$$

over a bounded region Ω. Assume that u is the solution which is continuous, has continuous derivatives up to the second order and exists on the boundary of Ω. We will vary u by an arbitrary function $\eta(x,y)$ which $\eta = 0$ on the boundary curve of Ω and define the function u_ε by the equation

$$u_\varepsilon(x,y) = u(x,y) + \varepsilon\eta(x,y),$$

where ε is a parameter.

Because of arbitrariness of η, u_ε represents any function with continuous second derivatives on Ω. Among these u_ε, we want to pick the one function that makes *I* an extremum. Now *I* is a function of parameter ε. In the case $\varepsilon = 0$, we have

$$u_\varepsilon(x,y) = u(x,y),$$

which is the desired solution. Our problem is to let *I* having an extremum value wherever $\varepsilon = 0$. In other words, we want

$$\frac{dI}{d\varepsilon}(u+\varepsilon\eta)\,|_{\varepsilon=0} = 0.$$

Differentiating with respect to ε, we get

$$\frac{dI}{d\varepsilon}(u+\varepsilon\eta)\,|_{\varepsilon=0} = \iint_\Omega (F_u\eta + F_{u_x}\eta_x + F_{u_y}\eta_y)dxdy = 0,$$

which will be transformed by integrating by parts. This implies that

$$\frac{dI}{d\varepsilon}(u+\varepsilon\eta)\,|_{\varepsilon=0} = \iint_\Omega \eta(F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y})dxdy + \int_\Upsilon \eta F_{u_x}dy + \int_\Upsilon \eta F_{u_y}dy$$

where is the boundary Υ curve of Ω. Since $\eta = 0$ on the boundary Υ curve of Ω implies that the second term and the third term on the right side are disappear, by our objective, hence

$$\iint_\Omega \eta(F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y})dxdy = 0.$$

By continuity and arbitrariness of η on Ω, then

$$\frac{\partial}{\partial x}F_{u_x} + \frac{\partial}{\partial y}F_{u_y} - F_u = 0,$$

which is called Euler-Lagrange differential equation. In addition, there are quite a lot of works establishing the existence of extremum and characterizing them. In many cases, extremal functions or curves can be expressed as solutions to differential equations. In 1992, Rudin, Osher, and Fatemi presented the mathematical denoising model which is called the ROF model based on COV. The ROF model considers *u* as a solution to a problem of COV which minimizes the functional

$$F(u) = \iint_{\Omega} \sqrt{u_x^2 + u_y^2}\, dxdy + \lambda \iint_{\Omega} (u - \tilde{u})^2\, dxdy,$$

where *u* is the desired image, $\tilde{u}$ is the observed image, Ω is the image domain and λ is a chosen parameter. By calculus of variations, the solution of this problem is satisfied if it is a solution of the Euler-Lagrange differential equation

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}}\right) + \lambda(u - \tilde{u}) = 0.$$

3. DESCRIPTION OF PROPOSED MODEL

Assume that $\tilde{u}$ is a given ultrasound image defined on Ω, an bounded rectangle, open subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. We assume $\tilde{u}$ is bounded and positive. We note that we will assume that $\tilde{u}$ has no noise on the boundary of Ω. Recall that the noise in ultrasound image is the speckle noise. The speckle noise is represented by

$$P(r) = \frac{r}{\sigma^2} e^{-r^2/\left(2\sigma^2\right)}.$$

Our discussion follows the ROF model. We wish to determine the image *u* that is most similar the given observed image $\tilde{u}$. The conditional probability shows

$$P(u \cap \tilde{u}) = P(\tilde{u} \mid u)P(u).$$

Thus, we wish to maximize $P(\tilde{u} \mid u)P(u).$ Since speckle noise is described by the Rayleigh distribution, we assume that region Ω has been pixelated by x_i, where $i = 1, \ldots, n$ and the values of $\tilde{u}$ at the pixels are independent. Then

$$P(\tilde{u} \mid u) = \left(\frac{\tilde{u}(x_1)}{\left(u(x_1)\right)^2} e^{-\frac{\left(\tilde{u}(x_1)\right)^2}{2\left(\tilde{u}(x_1)\right)^2}}\right) \cdots \left(\frac{\tilde{u}(x_n)}{\left(u(x_n)\right)^2} e^{-\frac{\left(\tilde{u}(x_n)\right)^2}{2\left(\tilde{u}(x_n)\right)^2}}\right).$$

The probability $P(u)$ comes from the assumption that

$$P(u) = e^{-\iint_\Omega \sqrt{u_x^2 + u_y^2}\,dxdy}.$$

Instead of maximizing $P(\tilde{u} \mid u)P(u),$ we minimize $-\ln P(\tilde{u} \mid u)P(u)$ because the natural logarithm function is an increasing function. The result is to seek a minimizer of

$$\sum_{i=1}^{n}\left(\frac{(\tilde{u}(x_i))^2}{2(u(x_i))^2} + 2\ln u(x_i) - \ln \tilde{u}(x_i)\right) + \iint_\Omega \sqrt{u_x^2 + u_y^2}\,dxdy,$$

which is equivalent a minimizer of

$$\iint_\Omega \sqrt{u_x^2 + u_y^2}\,dxdy + \sum_{i=1}^{n}\left(\frac{(\tilde{u}(x_i))^2}{(u(x_i))^2} + \ln u(x_i)\right).$$

We regard this as a discrete approximation of the functional

$$E(u) = \iint_\Omega \left(\sqrt{u_x^2 + u_y^2} + \frac{\tilde{u}^2}{u^2} + \ln u\right)dxdy.$$

Similarly, by COV the Euler-Lagrange equation for minimizing $E(u)$ is

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}}\right) + \left(\frac{2\tilde{u}^2}{u^3} - \frac{1}{u}\right) = 0.$$

4. NUMERICAL METHOD

The solution procedure uses the gradient descent method. This means that the numerical method is as follows. We let:

$$i = 0, \ldots, M,$$
$$j = 0, \ldots, N,$$
$$n = 0, 1, \ldots,$$
$$u_{ij}^0 = \tilde{u}_{ij}.$$

The numerical approximation is

$$u_{ij}^{n+1} = u_{ij}^{n} + \frac{\Delta t}{h}\left[\triangle_{-}^{x}\left(\frac{\triangle_{+}^{x} u_{ij}^{n}}{\sqrt{\left(\triangle_{+}^{x} u_{ij}^{n}\right)^{2} + \left(m(\triangle_{+}^{y} u_{ij}^{n}, \triangle_{-}^{y} u_{ij}^{n})\right)^{2}}}\right)\right]$$

$$+\frac{\Delta t}{h}\left[\triangle_{-}^{y}\left(\frac{\triangle_{+}^{y} u_{ij}^{n}}{\sqrt{\left(\triangle_{+}^{y} u_{ij}^{n}\right)^{2} + \left(m(\triangle_{+}^{x} u_{ij}^{n}, \triangle_{-}^{x} u_{ij}^{n})\right)^{2}}}\right)\right]$$

$$+\Delta t\left[\frac{2\left(\tilde{u}_{ij}\right)^{2}}{\left(u_{ij}^{n}\right)^{3}} - \frac{1}{\left(u_{ij}^{n}\right)}\right].$$

Here

$$\triangle_{\pm}^{x} u_{ij} = \pm\left(u_{(i\pm1)j} - u_{ij}\right)$$

and similarly,

$$\triangle_{\pm}^{y} u_{ij} = \pm\left(u_{i(j\pm1)} - u_{ij}\right).$$

The function *m* is defined by:

$$m(a,b) = \left(\frac{\operatorname{sgn}(a) + \operatorname{sgn}(b)}{2}\right)\min(|a|,|b|).$$

Note that a step size restriction is imposed for stability: $\frac{\Delta t}{h} \leq 1$.

RESULTS

To show that our results are be believable, we use the 18-percent grayscale image which is a standard image for color balance checking in our experiment, then we compute the correlation coeffient of the original image and the noisy image compare with the correlation coeffient of the original image and the reconstructed imag. Furthermore, we use Lenna image which is a well-known image in image processing field in our experiment. Similarly, we compute the correlation coeffient of the original image and the noisy image compare with the correlation coeffient of the original image and the reconstructed imag. We found that after enhance the images by the numerical process, the correlation coefficient of the original image and reconstructed images are greater than the correlation coefficient of the original image and noisy images. This show that the original image and the reconstructed image have better relationship and our result is claimed. The results show that the noise can be removed by the numerical process. The output of the prototype software shows that after enhance the ultrasound image, it has smoother image as present in figure 3.

figure 1a.
Original
18-percent
grayscale image.

figure 1b.
Noisy 18-percent
grayscale image
(0.001-variance
speckle).
Correlation Coefficient
of the original image
and this image is
0.0005

figure 1c.
Reconstructed image
(40-loops iterative
process).Correlation
Coefficient of the
original image and this
image is
0.0045

figure 2a.
Original
Lenna image
(www.math.tau.ac.il/
~turkel/lenna.jpg).

figure 2b.
Noisy
Lenna image
(0.01-variance
speckle).
Correlation
Coefficient of the
original image and
this image is 0.9836

figure 2c.
Reconstructed
Lenna image
(15-loops iterative
process).
Correlation Coefficient
of the original image
and this image is
0.9927

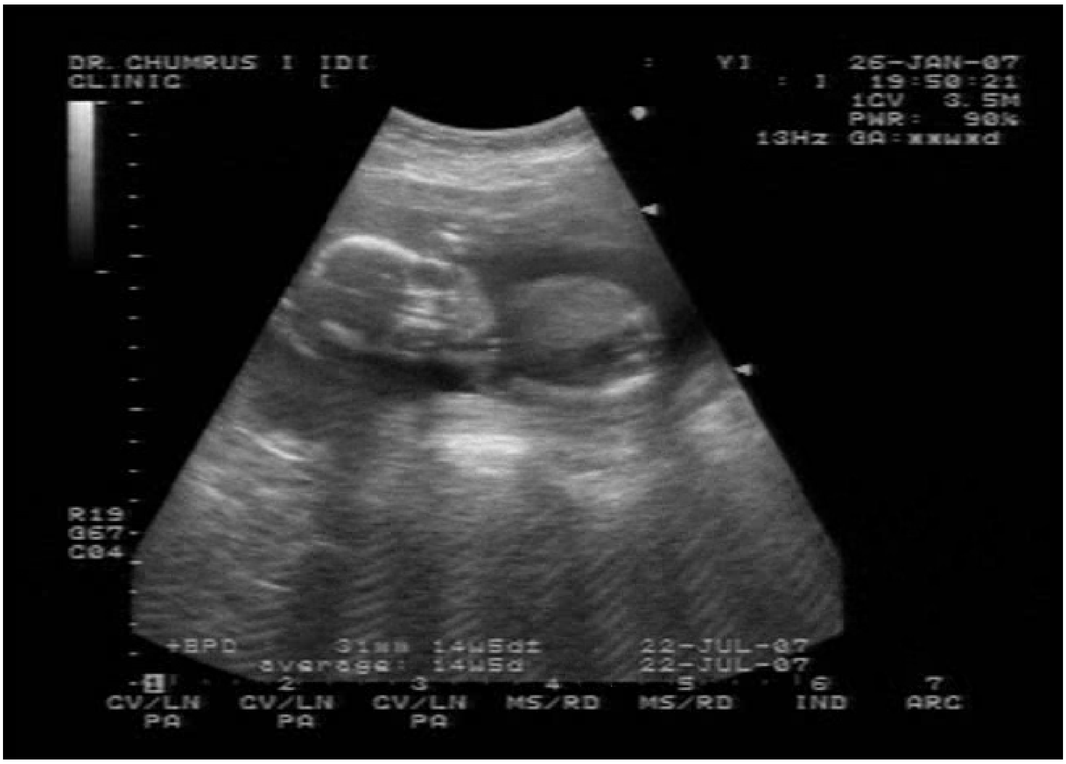

figure 3a. Original ultrasound image (Provided by Dr.Chumrus Sakulpisarn)

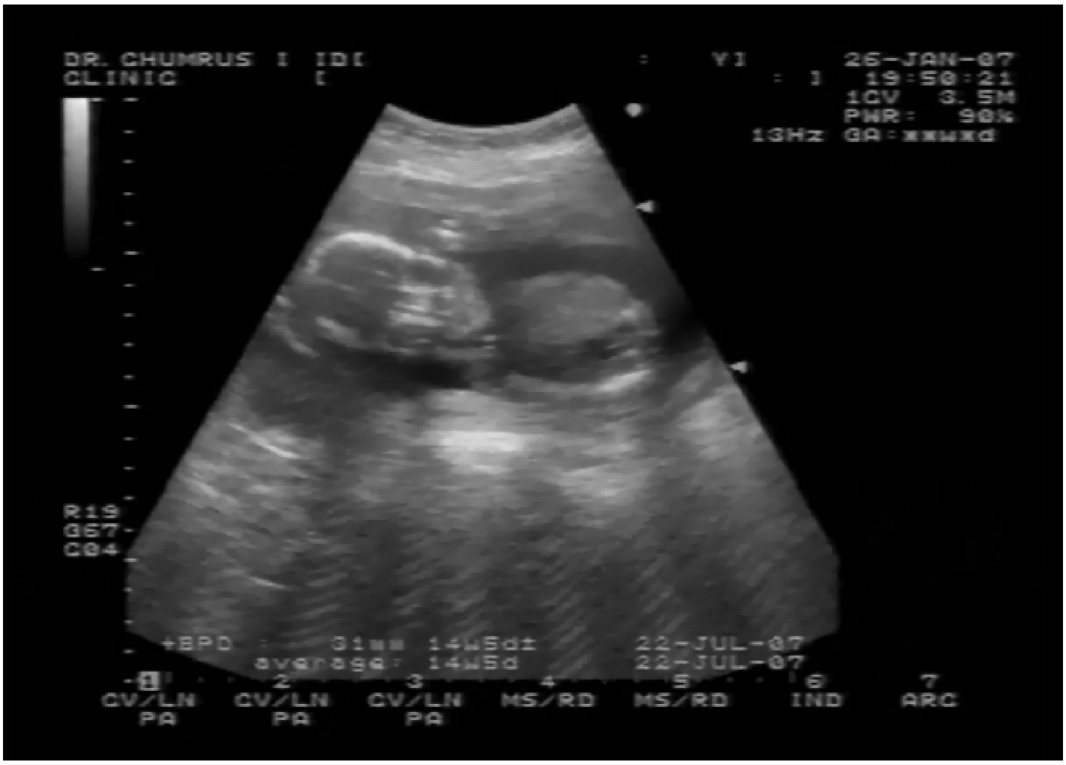

figure 3b. 3-loops iterative process reconstructed ultrasound image

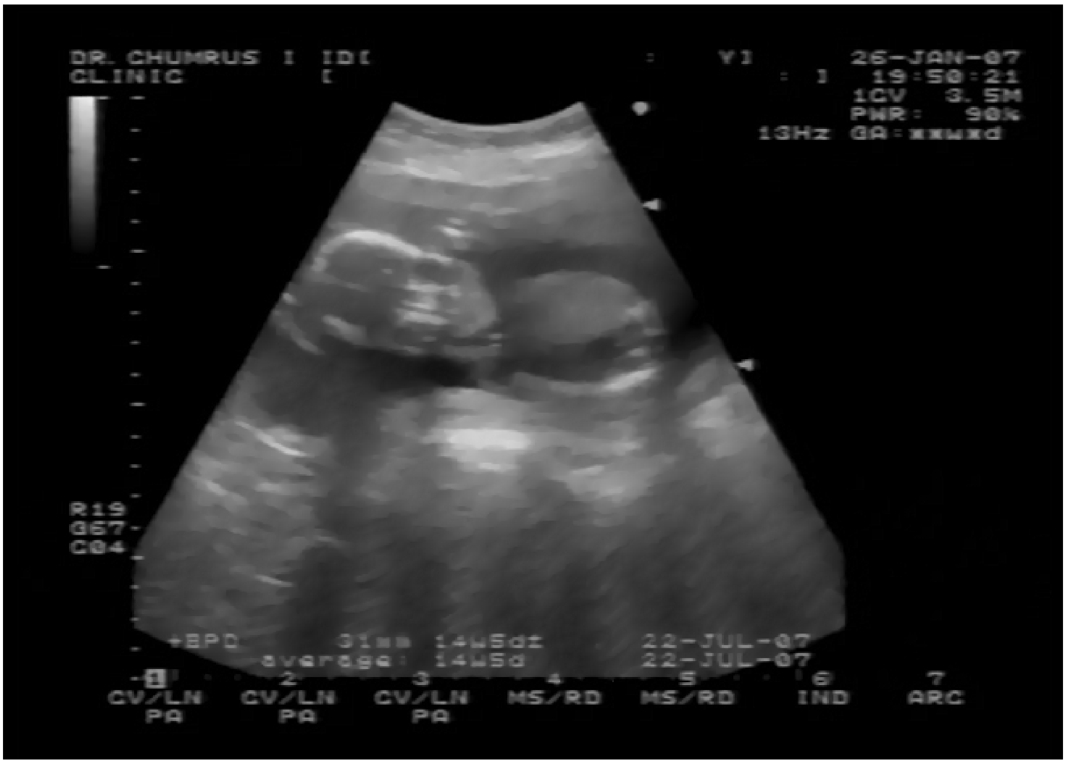

figure 3c. 5-loops iterative process reconstructed ultrasound image

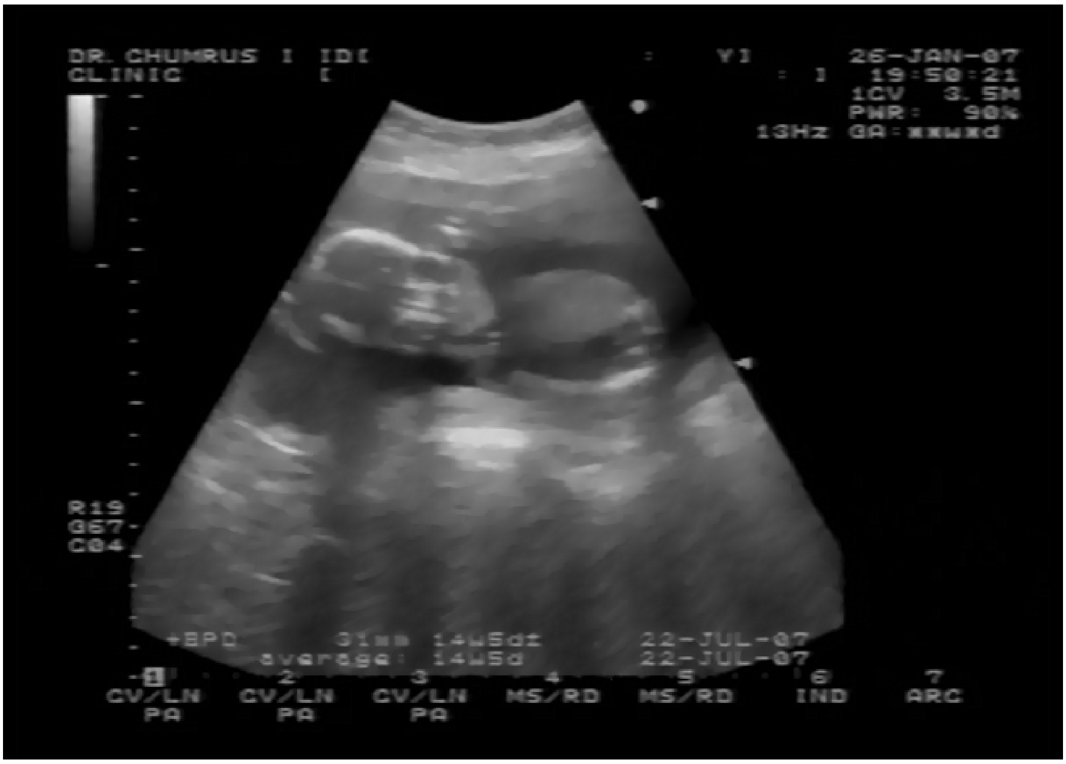

figure 3d. 9-loops iterative process reconstructed ultrasound image

CONCLUSION AND DISCUSSION

A new model for speckle reduction of ultrasound images is presented. The model is based on COV which leads to the Euler-Lagrange differential equation and approximate the solution by gradient descent method. Our technique can reduce the noise. However, the problem is the speed of process time. Moreover, because of we work with the grayscale 256 bits mode, thus the error may come from the round function in software implementation.

ACKNOWLEDGEMฎNTS

I would like to express my sincere gratitude to my thesis advisor, Asst.Prof. Dr. Jessada Tanthanuch and thesis co-advisor, Dr.Paramet Horkaew, for their support, patient help, providing me with extensive literature background and offering many useful suggestions. Also, I am indebted Dr.Chumrus Sakulpisarn for his ultrasound videos. In addition, I would like to acknowledge the personal and professional support received from the faculty of the School of Mathematics, Suranaree University of Technology: Assoc. Prof. Dr. Prapasri Asawakun, Prof. Dr. Sergey Meleshko, Prof. Dr. Pairote Sattayatham, Assoc. Prof. Dr. Nikolay Moshkin, Asst. Prof. Dr. Arjuna Chaiyasena, and Asst.Prof. Dr. Eckart Schulz

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