ANALYSIS ON $\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau),u(x,t)\right)$

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Abstract. *Equation* $\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau),u(x,t)\right)$ *is a delay partial differential equation with an arbitrary functional* G. *This delay partial differential equation is more general than* $\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau)\right)$ *which has been applied group analysis to find representations of analytical solutions* [3]. *Application of group analysis to the equation and group classification of representations of solutions where* $G=g\left(u(x,t)-u(x,t-\tau)\right)+H(u)$, g *and* H *are arbitrary functions, are presented in the article.*

1 Introduction

Consider delay partial differential equation (DPDE) with delay $\tau>0$

(1.1) $$\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau),u(x,t)\right)$$

For the simplicity, notation u^τ will be used to denote $u(x,t-\tau)$, u denotes $u(x,t)$ and u_x, u_t mean first partial derivatives of u with respect to x and t, respectively. Equation (1.1) can be simply written as

(1.2) $$u_t+uu_x=G\left(u^\tau,u\right).$$

Equation (1.2) is similar to Hopf or inviscid Burgers' equation [1]. However, equation (1.2) has a delay term, which makes the equation difficult to be solved [2]. Applications of delay differential equations can be found in [2,3,4,5,6]. The representations of solutions for the particular case of equation (1.2),

(1.3) $$u_t+uu_x=G\left(u^\tau\right).$$

has been found [3]. These solutions were obtained by applying group analysis method [7,8,9] to the equation. Group analysis also classifies equation (1.3) w.r.t. symmetries into two cases, arbitrary functional G and $G=ku^\tau$:

- **For arbitrary functional** $G(u^\tau)$

  The solution is $u=f_1(C_2x-C_1t)$, where f_1 is an arbitrary function, C_1,C_2 are arbitrary constants. The solution reduces equation (1.3) into a functional ordinary differential equation (FODE) $f_1'(\theta)=\frac{G\left(f_1(\theta+C_1\tau)\right)}{C_2f_1(\theta)-C_1}$, where $\theta=C_2x-C_1t$.

- **For particular functional** $G(u^\tau)=ku^\tau$, where *k* is an arbitrary constant.

For this case, equation (1.3) has two possible forms of representation of solutions, i.e.

1. $u=(x+C_3)f_2(t)$, where f_2 is an arbitrary function, C_3 is an arbitrary constant.

This solution reduces the equation into delay ordinary differential equation (DODE) $f_2'(\theta)=kf_2(t-\tau)-\left[f_2(t)\right]^2$.

2. $u=e^{C_5t}f_3\left((x+C_4)e^{-C_5t}\right)$, where f_2 is an arbitrary function, C_4,C_5 are arbitrary constants.

By this solution, equation (1.3) can be simplified to FODE $f_3'(\theta)=\frac{C_5f_3(\phi)-kf_3\left(e^{C_5\tau}\phi\right)}{f_3(\phi)-C_5\phi}$, where $\phi=(x+C_4)e^{-C_5t}$.

In this article, group analysis is applied to find symmetries of equation (1.2) which is more general than (1.3). However, for the sake of simplicity, equation (1.2) is considered for the case $G=g\left(u(x,t)-u(x,t-\tau)\right)+H(u)$ only, where g and H are arbitrary functions. Classification of the equation with respect to groups of symmetries admitted by the equation are presented in the following sections.

2 Applications of group analysis to delay differential equations

By the theory of group analysis, a *symmetry* of equation (1.2) is defined as the transformation $\varphi:\Omega\times\Delta\to\Omega$ which transforms a solution of the differential equation to a solution of the same equation, where Ω is a set of variables (x,t,u) and $\Delta\subset\mathbb{R}$ is a symmetric interval with respect to zero. Variable ε is considered as a parameter of transformation φ, which transforms variable (x,t,u) to new variable $(\overline{x},\overline{t},\overline{u})$ of the same space. Let $\varphi(x,t,u;\varepsilon)$ be denoted by $\varphi_\varepsilon(x,t,u)$. The set of functions φ_ε forms *a one-parameter transformation group of space* Ω if the following properties hold [7,8,9]:

(1) $\varphi_0(x,t,u)=(x,t,u)$ for any $(x,t,u)\in\Omega$;

(2) $\varphi_{\varepsilon_1}\left(\varphi_{\varepsilon_2}(x,t,u)\right)=\varphi_{\varepsilon_1+\varepsilon_2}(x,t,u)$ for any $\varepsilon_1,\varepsilon_2,\varepsilon_1+\varepsilon_2\in\Delta$ and $(x,t,u)\in\Omega$;

(3) if $\varphi_\varepsilon(x,t,u)=(x,t,u)$ for any $(x,t,u)\in\Omega$, then $\varepsilon=0$.

The other notations $\overline{x}=\varphi^x(x,t,u;\varepsilon),\overline{t}=\varphi^t(x,t,u;\varepsilon),\overline{u}=\varphi^u(x,t,u;\varepsilon)$ are used as the same meaning as $\varphi_\varepsilon(x,t,u)=(\overline{x},\overline{t},\overline{u})$. The transformed variable u with delay term and its derivatives are defined by $\overline{u}^\tau=\overline{u}(\overline{x},\overline{t}-\tau)$ and $\overline{u}_{\overline{x}}=\partial\overline{u}/\partial\overline{x},\overline{u}_{\overline{t}}=\partial\overline{u}/\partial\overline{t}$, respectively. Consider DPDE

(2.1) $$F(x,t,u,u^\tau,u_x,u_t)=0.$$

[6] shows the derivative of an equation, with the transformed variables $\overline{x},\overline{t},\overline{u}$ and derivatives $\overline{u}_{\overline{x}},\overline{u}_{\overline{t}}$, with respect to parameter ε vanishes if the transformation is symmetry :

(2.2) $$\left.\frac{\partial F(\overline{x},\overline{t},\overline{u},\overline{u}^\tau,\overline{u}_{\overline{x}},\overline{u}_{\overline{t}})}{\partial\varepsilon}\right|_{\varepsilon=0,(2.1)}=\left.\widetilde{X}F(x,t,u,u^\tau,u_x,u_t)\right|_{(2.1)}\equiv 0.$$

The operator $\widetilde{X}$ is defined by $\widetilde{X}=(\zeta-u_x\xi-u_t\eta)\partial_u+(\zeta^\tau-u_x^\tau\xi^\tau-u_t^\tau\eta^\tau)\partial_{u^\tau}+\zeta^{u_x}\partial_{u_x}+\zeta^{u_t}\partial_{u_t}$, where

$$\xi(x,t,u)=\frac{\partial\varphi^x}{\partial\varepsilon}(x,t,u;0),\eta(x,t,u)=\frac{\partial\varphi^t}{\partial\varepsilon}(x,t,u;0),\zeta(x,t,u)=\frac{\partial\varphi^u}{\partial\varepsilon}(x,t,u;0),$$
$$\xi^\tau=\xi(x,t-\tau,u^\tau),\eta^\tau=\eta(x,t-\tau,u^\tau),\zeta^\tau=\zeta(x,t-\tau,u^\tau),$$
$$\zeta^{u_x}=D_x(\zeta-u_x\xi-u_t\eta),\zeta^{u_t}=D_t(\zeta-u_x\xi-u_t\eta),$$
$$D_x=\partial_x+u_x\partial_u+u_x^\tau\partial_{u^\tau}+u_{xx}\partial_{u_x}+u_{xt}\partial_{u_t}+\ldots,$$
$$D_t=\partial_t+u_t\partial_u+u_t^\tau\partial_{u^\tau}+u_{xt}\partial_{u_x}+u_{tt}\partial_{u_t}+\ldots,$$

The operator $\widetilde{X}$ is is called *a canonical Lie-Bäcklund infinitesimal generator* of a symmetry. Equation (2.2) is called *a determining equation* (DME). Since $\left.\widetilde{X}F(x,t,u,u^\tau,u_x,u_t)\right|_{(2.1)}\equiv 0$, we say that the operator $\widetilde{X}$ is *admitted* by equation (2.1) or equation (2.1) *admits* the operator $\widetilde{X}$. Lie's theory [7,8,9] shows the generator is one-to-one correspondent to the symmetry. This generator is also equivalent to *an infinitesimal generator* [8]

(2.3) $$X=\xi\partial_x+\eta\partial_t+\zeta\partial_u.$$

3 Finding and solving the determining equation

The DME for $u_t+uu_x=G(u^\tau,u)$ can be found by letting $F=u_t+uu_x-G(u^\tau,u)$ and substitute it into equation (2.2),

(3.1) $$\left.\widetilde{X}\left(u_t+uu_x-G(u^\tau,u)\right)\right|_{u_t=G(u^\tau,u)-uu_x}\equiv 0.$$

By letting $u_t=G-uu_x$ so $u_{xt}=u_xG_u+u_x^\tau G_{u^\tau}-(u_x)^2-uu_{xx},\ u_{tt}=u_tG_u+u_t^\tau G_{u^\tau}-u_xu_t-uu_{xt}$ and $u_t^\tau=G^\tau-u^\tau u_x^\tau$, where

where $G^\tau=G(u^\tau,u^{\tau\tau}),\ u^{\tau\tau}=u(x,t-2\tau),\ u_t^\tau=u_t(x,t-\tau),\ u_x^\tau=u_x(x,t-\tau)$. Thus DME (3.1) becomes

(3.2) $$\begin{gathered}u_x^\tau G_{u^\tau}\left[u^\tau(\eta-\eta^\tau)+\xi^\tau-\xi\right]+u_x\left[u(\eta_t+\eta_xu+\eta_uG)-(\xi_t+\xi_xu+\xi_uG)+\zeta\right]\\-G(\eta_t+\eta_xu+\eta_uG)+G_{u^\tau}\left[G^\tau(\eta^\tau-\eta)-\zeta^\tau\right]-G_u\zeta+\zeta_t+\zeta_xu+\zeta_uG\equiv 0\end{gathered}.$$

By the theory of existence of a solution of a delay differential equation, the initial value problem has a particular solution corresponding to a particular initial value. Because initial values are arbitrary, variables u, u^τ and their derivatives can be considered as arbitrary elements. Since every transformed-solution $\overline{u}(\overline{x},\overline{t})$ is a solution of equation (2.1), the DME must be identical to zero. Thus, if DME (2.2) is written as a polynomial of variables and their derivatives, the coefficients of these variables in the equations must vanish. In order to solve a DME, one solves the several equations of these coefficients. This method is called *splitting the DME*. Unknown functions ξ, η and ζ can be obtained from this process.

By splitting equation (3.2) with respect to u_x^τ, one obtains $G_{u^\tau}\left[u^\tau(\eta-\eta^\tau)+\xi^\tau-\xi\right]\equiv 0$. Since equation (1.2) is considered as a DPDE, it is assumed $G_{u^\tau}\neq 0$. The equation is simplified to

(3.3) $$u^\tau(\eta-\eta^\tau)+\xi^\tau-\xi\equiv 0$$

By the assumption ξ and η depend on variables x,t,u while ξ^τ and η^τ depend on x,t,u^τ, if one differentiates equation (3.3) w.r.t. u, the derivative becomes $u^\tau\eta_u-\xi_u\equiv 0$. Splitting the equation w.r.t u^τ implies $\xi_u=0$ and $\eta_u=0$, which means ξ and η do not depend on u. By the similar structure of ξ^τ and ξ, and η^τ and η, both ξ^τ and η^τ depend on only variables x and t. Equation (3.3) can be split again w.r.t. u^τ which implies $\xi^\tau(x,t)=\xi(x,t)$ and $\eta^\tau(x,t)=\eta(x,t)$. The conditions obtained mean ξ and η are periodic functions w.r.t. t with period τ, i.e.

(3.4) $$\xi(x,t-\tau)=\xi(x,t),\quad \eta(x,t-\tau)=\eta(x,t).$$

Again, splitting the DME w.r.t. u_x, one gets

(3.5) $$\zeta=\xi_t+\xi_x u-u(\eta_t+\eta_x u),\quad \zeta^\tau=\xi_t+\xi_x u^\tau-u^\tau(\eta_t+\eta_x u^\tau).$$

Substitute ξ,η,ζ and ζ^τ into the DME and differentiate it with respect to u^τ,

(3.6) $$\begin{aligned}&\xi_t\left(-[G_{uu^\tau}+G_{u^\tau u^\tau}]\right)+\xi_x\left(-[G_{uu^\tau}u+G_{u^\tau u^\tau}u^\tau]\right)+\\&\eta_t\left(G_{uu^\tau}u+G_{u^\tau u^\tau}u^\tau-G_{u^\tau}\right)+\eta_x\left(G_{uu^\tau}u^2+G_{u^\tau u^\tau}(u^\tau)^2-3G_{u^\tau}u+2G_{u^\tau}u^\tau\right)\equiv 0\end{aligned}$$

Here if we consider equation (3.6) as $\xi_t\mathrm{A}+\xi_x\mathrm{B}+\eta_t\mathrm{C}+\eta_x\mathrm{D}\equiv 0$, which may be written in a vector form as

(3.7) $$\langle\xi_t,\xi_x,\eta_t,\eta_x\rangle\cdot\langle\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D}\rangle\equiv 0,$$

where $\mathrm{A}=-[G_{uu^\tau}+G_{u^\tau u^\tau}]$, $\mathrm{B}=-[G_{uu^\tau}u+G_{u^\tau u^\tau}u^\tau]$, $\mathrm{C}=G_{uu^\tau}u+G_{u^\tau u^\tau}u^\tau-G_{u^\tau}$ and $\mathrm{D}=G_{uu^\tau}u^2+G_{u^\tau u^\tau}(u^\tau)^2-3G_{u^\tau}u+2G_{u^\tau}u^\tau$, we are able to classify equation (1.2) as the followings.

3.1 The kernel of admitted Lie groups

The set of symmetries, which are admitted for any functional appeared in the equation is called a *kernel* of admitted generators. Assume equation (3.6) is valid for any functional G. Since G, G_{u^τ}, G_{uu^τ}, $G_{u^\tau u^\tau}$ vary arbitrarily, the set spanned by $\langle\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D}\rangle$ has dimension 4. Thus $\langle\xi_t,\xi_x,\eta_t,\eta_x\rangle$ must be a zero vector, i.e. all of $\xi_t,\xi_x,\eta_t,\eta_x$ vanish. This implies ξ and η are constants and ζ is zero. Let ξ and η be denoted by C_1 and C_2, respectively. The infinitesimal generator admitted by equation (1.2) is $X=C_1\partial_x+C_2\partial_t$. By the theory from group analysis, the characteristic equations $\frac{dx}{\xi}=\frac{dt}{\eta}=\frac{du}{\zeta}$ imply $u=f(C_2x-C_1t)$ is a representation of a solution. It reduces equation (1.2) into FODE

$$f'(\theta)=\frac{G\left(f(\theta),f(\theta+C_1\tau)\right)}{C_2f(\theta)-C_1},$$

where $\theta=C_2x-C_1t$.

3.2 Extension of the kernel

Extensions are symmetries for the particular functional G. Here, for the sake of simplicity, case $\mathrm{A}=0$ is considered only. For this case, it implies

(3.8) $$G(u,u^\tau)=g(u-u^\tau)+H(u),$$

where g is an arbitrary function of $u-u^\tau$ such that $\frac{\partial g}{\partial u^\tau}\neq 0$ (or $g'\neq 0$) and H is an arbitrary function of variable u. Equation (3.6) is reduced into the form

(3.9) $$\xi_x(u-u^\tau)g''+\eta_t\left(g'-[u-u^\tau]g''\right)+\eta_x\left(-\left[u^2-(u^\tau)^2\right]g''+[3u-2u^\tau]g'\right)\equiv 0.$$

Equation (3.9) can be considered as a vector form $\langle\xi_x,\eta_t,\eta_x\rangle\cdot\langle A,B,C\rangle\equiv 0$, where $A=(u-u^\tau)g''$, $B=g'-[u-u^\tau]g''$ and $C=-\left[u^2-(u^\tau)^2\right]g''+[3u-2u^\tau]g'$. Let $\mathbb{V}$ be the set spanned by vector $\langle A,B,C\rangle$. All possible cases which make equation (3.9) valid are considered according to the dimension of $\mathbb{V}$.

3.2.1 **dim $\mathbb{V}$ =3.** This condition means vector $\langle \xi_x, \eta_t, \eta_x \rangle$ must be a zero vector, i.e. ξ_x, η_t, η_x vanish. Thus the DME is simplified to $-H'(u)\xi'(t) + \xi''(t) = 0$. The derivative of the DME w.r.t. u is $-H''(u)\xi'(t) = 0$.

- **Case** $H''(u) = 0$, Here $H(u) = H_1 u + H_2$ is a solution of the equation, where H_1, H_2 are arbitrary constants. However, by the arbitrariness of function g, H_2 can be omitted. The DME is $-H_1\xi'(t) + \xi''(t) = 0$, which has $\xi = C_1 e^{H_1 t} + C_2$ as a solution, where C_1, C_2 are arbitrary constants. The periodic condition (3.4) of ξ implies $C_1 e^{H_1(t-\tau)} + C_2 = C_1 e^{H_1 t} + C_2$. The condition is valid for $H_1 = 0$ or $C_1 = 0$. For this case ξ must be a constant.

- **Case** $H''(u) \neq 0$. The equation immediately implies ξ is a constant.

Both two cases show equation $G(u, u^\tau) = g(u - u^\tau) + H(u)$ admits $X = \xi\partial_x + \eta\partial_t$, where ξ and η are arbitrary constants and g, H are arbitrary functions. For **dim $\mathbb{V}$ =3**, it has the same solution with the kernel case.

3.2.2 **dim $\mathbb{V}$ =2.** This condition means there exists a constant vector $\langle \alpha, \beta, \gamma \rangle \neq 0$ which is orthogonal to set $\mathbb{V}$, i.e. $\langle \alpha, \beta, \gamma \rangle \cdot \langle A, B, C \rangle = \alpha A + \beta B + \gamma C = 0$. By changing of variable $z = (u - u^\tau)$, the equation is derived to $z(\alpha - \beta + \gamma z)g'' + (\beta + 3\gamma z)g' + u^\tau \gamma(-2zg'' + g') = 0$. Splitting the equation w.r.t. u^τ, we have

(3.10) $$\gamma(-2zg'' + g') = 0,$$

(3.11) $$z(\alpha - \beta + \gamma z)g'' + (\beta + 3\gamma z)g' = 0.$$

- **Case** $\gamma \neq 0$. Solving equation (3.10) makes $g(z) = C_1 z^{3/2} + C_2$, where C_1, C_2 are arbitrary constants. Equation (3.11) is simplified to $\frac{3}{4}C_1\left([\alpha + \beta]\sqrt{z} + 5\gamma z^{3/2}\right) = 0$. By the arbitrariness of z, $\alpha + \beta$ and γ must vanish. This case contradicts to the assumption $\gamma \neq 0$.

- **Case** $\gamma = 0$. Equation (3.15) is reduced to

(3.12) $$z(\alpha - \beta)g'' + \beta g' = 0$$

If $\alpha - \beta = 0$ (or $\alpha = \beta$) it makes $\beta g' = 0$. This case contradicts to the condition $\langle \alpha, \beta, \gamma \rangle \neq 0$ is not zero and $g' \neq 0$. Condition $\alpha - \beta \neq 0$ (or $\alpha \neq \beta$) will be considered only.

For the conditions $\gamma = 0$, $\alpha \neq \beta$, equation (3.12) is considered into two cases :

Case $\frac{\beta}{\alpha - \beta} = -1$, i.e. $\alpha = 2\beta$. The above condition $\alpha \neq \beta$ implies $\alpha \neq 0$. The equation can be reduced to $zg'' + g' = 0$, which has a solution $g(z) = C_1 \ln z + C_2$, where C_1 is a nonzero arbitrary constants, C_2 is a constant. However, the constant C_2 can be omitted because of the arbitrariness of H. Substitute g into the DME and differentiate it w.r.t. u^τ, the equation calculated is $\frac{C_1}{u - u^\tau}[2\eta_t - \xi_x + 4\eta_x u + \eta_x u^\tau] = 0$. Since $C_1 \neq 0$ and unknown functions ξ and η depend on (x, t), the equation can be split w.r.t. u and u^τ which implies $\eta = \eta(t)$ and $\xi(x,t) = 2\eta'(t)x + \xi_2(t)$, where ξ_2 is an arbitrary function of t. Substitute both obtained functions into the DME :

(3.13) $$2[\eta'''(t) - \eta''(t)H'(u)]x + [3\eta''(t) - \eta'(t)H'(u)]u - C_1\eta'(t) + \xi_2''(t) - H'(u)\xi_2'(t) = 0.$$

Since unknown functions ξ_2, η, H do not depend on x, then $\eta'''(t) - \eta''(t)H'(u) = 0$. This can be considered into subcases $H'(u) = 0$ and $H'(u) \neq 0$.

(1) $H'(u) = 0$, i.e. H is a constant. Then $\eta'''(t) = 0$. The periodic condition implies η is only a constant. The DME is simplified to $\xi_2''(t) = 0$. ξ_2 is also a constant by the periodic condition. This subcase shows $u_t + uu_x = C_1 \ln(u - u^\tau) + H$ admits the generator $\xi\partial_x + \eta\partial_t$ where H, ξ, η are arbitrary constants.

(2) $H'(u) \neq 0$. The mixed derivative of DME (3.13) w.r.t. x and u shows $-2\eta''(t)H''(u) = 0$. This can be considered into two subcases $H''(u) = 0$ and $H''(u) \neq 0$.

- $H''(u) = 0$. It implies $H = H_1 u + H_2$, where H_1, H_2 are arbitrary constants and $H_1 \neq 0$. The derivative of equation (3.13) w.r.t. x is $2\left(\eta'''(t) - H_1\eta''(t)\right) = 0$. Thus its solution is $\eta = C_3 + C_4 t + C_5 e^{H_1 t}$. By the periodic condition, C_4 and C_5 must identical to zero, i.e. η is a constant. The DME is reduced to $\xi_2''(t) - H_1\xi_2'(t) = 0$, which has a solution $\xi_2 = C_6 + C_7 e^{H_1 t}$. Also the periodic condition of ξ implies $C_7 = 0$. Then $u_t + uu_x = C_1 \ln(u - u^\tau) + H_1 u + H_2$ admits the generator $C_6\partial_x + C_3\partial_t$.

- $H''(u) \neq 0$. The equation implies $\eta''(t) = 0$, i.e. with the periodic condition η is a constant only. The DME is reduced to $\xi_2''(t) - H'(u)\xi_2'(t) = 0$. Differentiate the equation w.r.t. u, $-H''(u)\xi_2'(t) = 0$, it implies ξ_2 is a constant.

All above cases shows $u = f(\eta x - \xi t)$, where ξ, η are arbitrary constants and f is arbitrary function, is solution of $u_t + uu_x = C_1 \ln(u - u^\tau) + H(u)$, where C_1 is a nonzero arbitrary constant and H is an arbitrary function of u.

Case $\dfrac{\beta}{\alpha - \beta} \neq -1$. Let $\delta = \dfrac{\beta}{\alpha - \beta}$. Hence the solution of equation (3.12) is $g = C_1(u - u^\tau)^{\delta+1} + C_2$, where C_1 is a nonzero arbitrary constants, C_2 is a constant. However, the constant C_2 can be omitted because of the arbitrariness of H. Splitting the DME equation w.r.t. u^τ shows

(3.14) $$C_1(\delta+1)(u - u^\tau)^\delta \left[\delta\xi_x + (1-\delta)\eta_t + \left(3u - 2u^\tau - \delta(u + u^\tau)\right)\eta_t\right] = 0.$$

Since ξ and η depend on (x,t) then equation (3.14) can be split w.r.t. u and u^τ. It implies $(3-\delta)\eta_x = 0$ and $-(2+\delta)\eta_x = 0$. The arbitrariness of δ implies $\eta_x = 0$, i.e. $\eta = \eta(t)$. Equation (3.14) is simplified to

(3.15) $$\delta\xi_x + (1-\delta)\eta'(t) = 0.$$

Case $\delta \neq 0$.

(1) If $\delta = 1$, equation (3.15) shows $\xi_x = 0$, i.e. $\xi = \xi(t)$. The DME is reduced to

$$\xi''(t) - \eta''(t)u - 2\eta'(t)H(u) + [\eta'(t)u - \xi'(t)]H'(u) = 0.$$

The second derivative of DME w.r.t. u implies $[\eta'(t)u - \xi'(t)]H'''(u) = 0$.

If $H'''(u) \neq 0$, then $\eta'(t)u - \xi'(t) = 0$. Splitting the equation w.r.t. u shows ξ and η are constants. Thus the solution of equation $u_t + uu_x = C_1 \ln(u - u^\tau) + H(u)$ is also $u = f(\eta x - \xi t)$.

Suppose $H'''(u) = 0$. This means $H = H_1 u^2 + H_2 u + H_3$, where H_1, H_2, H_3 are arbitrary constants. The derivative of the DME w.r.t. u is

$$-[2H_1\xi'(t) + \eta''(t) + H_2\eta'(t)] = 0.$$

(a) $H_1 = 0$ **and** $H_2 = 0$. The periodic condition implies η is a constant. The DME is reduced to $\xi''(t) = 0$. So ξ is also a constant.

(b) $H_1 = 0$ **but** $H_2 \neq 0$. The equation shows $\eta = C_3 + C_4 e^{-H_2 t}$. The periodic condition reduces term $C_4 e^{-H_2 t}$ which makes η a constant. The DME is reduced to $\xi''(t) - H_2\xi(t) = 0$, also ξ must be a constant.

(c) $H_1 \neq 0$. Then $\xi'(t) = -\dfrac{\eta''(t) + H_2\eta'(t)}{2H_1}$. The DME is simplified to $\eta'''(t) + \lambda\eta'(t) = 0$, where $\lambda = 4H_1(C_1 + H_3) - (H_2)^2$.

If $\lambda = 0$, this shows $\eta'''(t) = 0$. ξ can be only a constant and ξ is a constant also.

If $\lambda < 0$, $\eta = C_3 + C_4 e^{\lambda t} + C_5 e^{-\lambda t}$. The periodic property of η implies C_4 and C_5 vanish and ξ must be also a constant.

If $\lambda > 0$, $\eta = C_3 + C_4 \cos\lambda t + C_5 \sin\lambda t$. By the periodic condition, it is considered into two cases :

- $\tau \neq \dfrac{2\pi}{\lambda}$. In this case, C_4 and C_5 must vanish and it implies ξ to be a constant.

- $\tau = \dfrac{2\pi}{\lambda}$. Here ξ is equal to $\eta = C_6 + C_7 \cos\lambda t + C_8 \sin\lambda t$, where C_6 is an arbitrary constant, $C_7 = -\dfrac{H_2 C_4 + \lambda C_5}{2H_1}, C_8 = -\dfrac{H_2 C_5 + \lambda C_4}{2H_1}$. By the condition (3.5),

$$\zeta = \lambda\left[(C_8 - C_5 u)\cos\lambda t + (-C_7 + C_4 u)\sin\lambda t\right]$$

The solution of equation $u_t + uu_x = C_1(u - u^\tau)^2 + H_1 u^2 + H_2 u + H_3$ can be found from the characteristic equations, i.e. $u = e^{-\int p\,dt}\left[\int q e^{\int p\,dt}\,dt + \mathfrak{F}(x - \psi(t))\right]$, where

$$p = \lambda\frac{C_5\cos\lambda t - C_4\sin\lambda t}{C_3 + C_4\cos\lambda t + C_5\sin\lambda t}, q = \lambda\frac{C_8\cos\lambda t - C_7\sin\lambda t}{C_3 + C_4\cos\lambda t + C_5\sin\lambda t}, \psi(t) = \int\frac{C_6 + C_7\cos\lambda t + C_8\sin\lambda t}{C_3 + C_4\cos\lambda t + C_5\sin\lambda t}dt,$$

$\mathfrak{F}$ is an arbitrary function, $C_1, C_3, C_4, C_5, C_6, H_1, H_2, H_3$ are arbitrary constants, $H_1 \neq 0$, λ, τ, C_7, C_8 are the constants which were defined in this section.

(2) If $\delta \neq 1$, equation (3.15) implies $\xi = \left(\frac{\delta-1}{\delta}\right)\eta'(t)x + \xi_2(t)$, where $\xi_2(t)$ is an arbitrary function of t. Substitute ξ into the DME and differentiate it w.r.t. both x and u, we obtain $\left(\frac{\delta-1}{\delta}\right)H''(u)\eta''(t) = 0$. This may be considered into two subcases.

(a) $H''(u) \neq 0$. This implies $\eta''(t) = 0$. Similar to the previous case, η is a constant. The DME is reduced to $\xi_2''(t) - H'(u)\xi_2'(t) = 0$. The derivative of the equation w.r.t. u shows $H''(u)\xi_2'(t) = 0$, which means ξ_2 is a constant. This shows both ξ and η are constants.

(b) $H''(u) = 0$. Then $H = H_1 u + H_2$, where H_1, H_2 are arbitrary constants. Derivative of the DME w.r.t. u is $\left(\frac{\delta-2}{\delta}\right)\eta''(t) - H_1\eta'(t) = 0$.

If $\delta = 2$, the derivative of the DME w.r.t. x gives us $\eta'''(t) - H_1\eta''(t) = 0$. Similar to the previous case, η is a constant. The DME is also $\xi_2''(t) - H_1\xi_2'(t) = 0$ and its solution is a constant.

If $\delta \neq 2$. For arbitrary constant H_1 and periodic property, η must be a constant. The DME is $\xi_2''(t) - H_1\xi_2'(t) = 0$ and its solution is a constant.

Both subcases show equation $u_t + uu_x = C_1(u - u^\tau)^{\delta+1} + H(u)$, for $\delta \neq -1$, has the same solution with the kernel case.

Case $\delta = 0$. Equation (3.15) shows $\eta'(t) = 0$, i.e. η is a constant. The DME is reduced to $H(u)\xi_x - H'(u)(\xi_t + u\xi_x) + u^2\xi_{xx} + 2u\xi_{xt} + \xi_{tt} = 0$. In order to classify a solution of DPDE, we have to analyze by the following cases :

(1) $\xi_x = 0$. The DME is simplified to $\xi''(t) - H'(u)\xi'(t) = 0$. Its derivative w.r.t. u is $-H''(u)\xi'(t) = 0$.

If $H''(u) = 0$, then $H = H_1 u + H_2$, where H_1, H_2 are arbitrary constants. The DME is $\xi''(t) - H_1\xi'(t) = 0$. With the periodic condition, ξ can be only a constant.

If $H''(u) \neq 0$, then $\xi'(t) = 0$ which show ξ is also a constant.

(2) $\xi_x \neq 0$. The third derivative of the DME w.r.t. u can be rewritten as

(3.16) $$\left(\frac{\xi_t}{\xi_x} + u\right)H^{(4)}(u) + 2H'''(u) = 0.$$

The derivatives of equation (3.16) w.r.t. x and t give $\frac{d}{dt}\left(\frac{\xi_t}{\xi_x}\right)H^{(4)}(u) = 0$ and $\frac{d}{dx}\left(\frac{\xi_t}{\xi_x}\right)H^{(4)}(u) = 0$. We consider the problem into two subcases :

\- $H^{(4)}(u) = 0$ then $H'''(u) = 0$ which makes $H = H_1u^2 + H_2u + H_3$ satisfying equation (3.16). The second derivative of the DME w.r.t. u is $2(\xi_{xx} - H_1\xi_x) = 0$.

If $H_1 = 0$ then $\xi(x,t) = \xi_1(t)x + \xi_2(t)$. The derivative of the DME w.r.t. u shows $2\xi_1'(t) = 0$, i.e. ξ_1 is a constant. The DME is $\xi_2''(t) - H_2\xi_2'(t) + H_3\xi_1 = 0$. Let $\lambda = (H_2)^2 - 4H_3\xi_1$. With the periodic condition, function ξ_2 can be found according to λ :

(a) $\lambda \geq 0$. ξ_2 must be a constant. Then the DME is $H_3\xi_1 = 0$. If $H_3 \neq 0$ then $\xi_1 = 0$. Thus $u_t + uu_x = C_1(u - u^\tau) + H_3$ admits the same generator and has the same solution with the kernel case. However, if $H_3 = 0$ and $\xi_1 \neq 0$ then $u_t + uu_x = C_1u + C_2u^\tau$, where C_1, C_2 are arbitrary constants, admits $(\xi_1x + \xi_2)\partial_x + \eta\partial_t + \xi_1u\partial_u$ and has a solution $u = (\xi_1x + \xi_2)\mathfrak{F}(\eta\ln(\xi_1x + \xi_2) - \xi_1t)$, where $\mathfrak{F}$ is an arbitrary function. This solution reduces the equation (3.17) into an FODE,

$$\mathfrak{F}'(\chi)[\eta\mathfrak{F}(\chi) - 1] + [\mathfrak{F}(\chi)]^2 = C_1\mathfrak{F}(\chi) + C_2\mathfrak{F}(\chi + \xi_1\tau)$$

where $\xi_1 \neq 0$, η are arbitrary constants and $\chi = \eta\ln(\xi_1x + \xi_2) - \xi_1t$.

(b) $\lambda < 0$. Let $\rho = \sqrt{-\lambda}/2$. Then $\xi_2 = e^{\frac{H_2}{2}t}(C_3\cos\rho t + C_4\sin\rho t)$. With the periodic condition, H_2 must be identical to zero, $H_3\xi_1 > 0$ and $\tau = \frac{2\pi}{\lambda}$. Then equation $u_t + uu_x = C_1(u - u^\tau) + H_3$, $H_3 \neq 0$, admits

$$(\xi_1x + C_3\cos\rho t + C_4\sin\rho t)\partial_x + \eta\partial_t + (\xi_1u - C_3\rho\sin\rho t + C_4\rho\cos\rho t)\partial_u,$$

where $\rho = \sqrt{H_3 \xi_1}, C_3, C_4$ are arbitrary constants. This case is too complicated to find an exact form of a solution.

If $H_1 \neq 0$ then $\xi(x,t) = \xi_1(t)e^{H_1 x} + \xi_2(t)$. The derivative of DME w.r.t. u is $-2H_1\xi_2'(t) = 0$ which means $\xi_2'(t) = 0$ or ξ_2 is a constant. DME is simplified to $e^{H_1 x}\left(\xi_1''(t) - H_2\xi_1'(t) + H_1H_3\xi_1(t)\right) = 0$. $\lambda = (H_2)^2 - 4H_1H_3$. With the periodic condition, function ξ_1 can be found according to λ :

(a) $\lambda \geq 0$. ξ_1 must be a constant.

If $H_2 > 0$, then $\xi_1 = 0$ and DME vanishes, i.e. the solution form is not different to the kernel case.

If $H_2 = 0$, then ξ_1 is any constant. The DME is $H_1H_3\xi_1 e^{H_1 x} = 0$. If $H_3 \neq 0$, the equation has the similar solution with the previous case. On the other hand, $H_3 = 0$ implies $u_t + uu_x = C_1(u - u^\tau) + H_1u^2$ admits $\left(\xi_1 e^{H_1 x} + \xi_2\right)\partial_x + \eta\partial_t + \xi_1 H_1 u e^{H_1 x}\partial_u$. This means $u = \left(\xi_1 e^{H_1 x} + \xi_2\right)\mathfrak{F}\left(\frac{\eta}{\xi_2 H_1}\ln\left(\frac{\xi_1 e^{H_1 x}}{\xi_1 e^{H_1 x} + \xi_2}\right) - t\right)$ is a solution of the equation and reduces the FPDE to $\mathfrak{F}'(\Theta) = C_1\frac{\mathfrak{F}(\Theta) - \mathfrak{F}(\Theta + \tau) + H_1\xi_2\left[\mathfrak{F}(\Theta)\right]^2}{\eta\mathfrak{F}(\Theta) - 1}$, where $\Theta = \frac{\eta}{\xi_2 H_1}\ln\left(\frac{\xi_1 e^{H_1 x}}{\xi_1 e^{H_1 x} + \xi_2}\right) - t$ and C_1 is an arbitrary constant.

(b) Let $\rho = \sqrt{-\lambda}/2$. Then $\xi_2 = e^{\frac{H_2}{2}t}\left(C_3\cos\rho t + C_4\sin\rho t\right)$. With the periodic condition, H_2 must be identical to zero, $H_1H_3 > 0$ and $\tau = \frac{2\pi}{\lambda}$. Then equation $u_t + uu_x = C_1(u - u^\tau) + H_1u^2 + H_3$, $H_3 \neq 0$, admits

$\left(e^{H_1 x}\left(C_3\cos\rho t + C_4\sin\rho t\right) + \xi_2\right)\partial_x + \eta\partial_t + e^{H_1 x}\left[H_1 u\left(C_3\cos\rho t + C_4\sin\rho t\right) + \rho\left(-C_3\sin\rho t + C_4\cos\rho t\right)\right]\partial_u$,

where $\rho = \sqrt{H_1H_3}$ and C_3, C_4 are arbitrary constants. This case is too complicated to find an exact form of a solution.

- $H^{(4)}(u) \neq 0$.It means ξ_t / ξ_x is a constant which has a solution $\xi = \psi(x + Kt)$, where K is a constant and ψ is an arbitrary function. Substitute ξ into equation (3.16), then $(K + u)H^{(4)}(u) + 2H'''(u) = 0$. The equation has a solution $H(u) = H_1(K + u)\ln(K + u) + H_2u^2 + H_3u + H_4$, where H_1, H_2, H_3, H_4 are constants. The DME is simplified to

$$u^2\left[\psi'' - H_2\psi'\right] + u\left[2k\psi'' - (H_1 + 2H_1K)\psi'\right] + K^2\psi'' + \left[H_4 - K(H_1 + H_3)\right]\psi' = 0.$$

Splitting the equation w.r.t. u^2 and u shows ψ is a constant. This case has the same solution with the kernel case.

3.2.3 **dim** $\mathbb{V}$ **=1.** Here $\langle A, B, C\rangle$ can be represented by $\langle A, B, C\rangle = \langle \alpha, \beta, \gamma\rangle\phi(u, u^\tau)$, where α, β, γ are arbitrary constants which $\langle \alpha, \beta, \gamma\rangle \neq \langle 0, 0, 0\rangle$ and ϕ is a nonconstant function. The system of equations corresponding to the vector is

(3.17) $$(u - u^\tau)g'' = \alpha\phi(u, u^\tau),$$

(3.18) $$g' - (u - u^\tau)g'' = \beta\phi(u, u^\tau),$$

(3.19) $$-\left[u^2 - (u^\tau)^2\right]g'' + 3ug' - 2u^\tau g' = \gamma\phi(u, u^\tau).$$

- Case $\alpha \neq 0$. Equation (3.19) can be derived from equation (3.17) and (3.18) into

$$\left[(2\alpha + 3\beta)u + (-3\alpha - 2\beta)u^\tau - \gamma\right]\phi = 0$$

Since ϕ is not identical to zero then its coefficient must vanish and implies $\alpha = \beta = \gamma = 0$. This contradicts to the assumption.

- Case $\alpha = 0$. Here $g'' = 0$, which implies $g = C_1(u - u^\tau) + C_2$, where C_1, C_2 are arbitrary constants. Substitute g into equation (3.18), $C_1 = \beta\phi(u, u^\tau)$ is obtained. If $\beta = 0$, it implies $C_1 = 0$ and g is a constant which is invalid. Also if β does not vanish, the equation implies ϕ is a constant function which also contradicts to the assumption.

This proves that case **dim** $\mathbb{V}$ **=1** is invalid.

3.2.4 **dim** $\mathbb{V}$ **=0.** $\langle A, B, C\rangle$ can be consider as a constant vector $\langle \alpha, \beta, \gamma\rangle$, i.e.

(3.20) $$(u - u^\tau)g'' = \alpha,$$

(3.21) $$g' - (u - u^\tau)g'' = \beta,$$

(3.22) $$-\left[u^2-(u^\tau)^2\right]g''+3ug'-2u^\tau g'=\gamma,$$

where α,β,γ are arbitrary constants. Substitute equation (3.20) into equation (3.21), it leads to $g'=\alpha+\beta$ and $g''=0$. Substitute both values into equation (3.22), the equation is reduced to

$$3(\alpha+\beta)u-2(\alpha+\beta)u^\tau=\gamma.$$

By the arbitrariness of u and u^τ, $\alpha+\beta$ vanishes which makes $g'=0$. It contradicts to the assumption. This case is invalid also.

4 Conclusion

Solutions of equation $u_t+uu_x=C_1(u-u^\tau)^2+H_1u^2+H_2u+H_3$, $u_t+uu_x=C_1u+C_2u^\tau$ $u_t+uu_x=C_1(u-u^\tau)+H_1u^2$ and $u_t+uu_x=C_1(u-u^\tau)+H_1u^2+H_3$ are presented in the article. For other forms of equation $u_t+uu_x=g(u-u^\tau)+H(u)$, where g,H are arbitrary functions, the solution is $u=f(\eta x-\xi t)$ where f is an arbitrary function and ξ,η are arbitrary constants.

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