Proceedings of the 2nd IMT-GT Regional Conference on Mathematics, Statistics and Applications
Universiti Sains Malaysia

SYMMETRY ANALYSIS ON $\dfrac{\partial u}{\partial t}(x,t) + u(x,t)\dfrac{\partial u}{\partial x}(x,t) = G\left(u(x,t-\tau)\right)$

Jessada Tanthanuch
School of Mathematics, Institute of Science, Suranaree University of Technology, Thailand
jessada@math.sut.ac.th

Abstract. Equation $\dfrac{\partial u}{\partial t}(x,t) + u(x,t)\dfrac{\partial u}{\partial x}(x,t) = G\left(u(x,t-\tau)\right)$ is a delay partial differential equation with an arbitrary functional G. Group analysis method is applied to find symmetries of the equation and to make group classification. Representations of analytical solutions and reduced equations are obtained from the symmetries.

1. Introduction

Consider delay partial differential equation with delay $\tau > 0$

$$\frac{\partial u}{\partial t}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = G\left(u(x,t-\tau)\right). \tag{1.1}$$

For simplicity, notation u^τ will be used to denote $u(x,t-\tau)$, u denote $u(x,t)$ and u_x, u_t mean first partial derivatives of u with respect to x and t, respectively. Equation (1.1) can be simply written as

$$u_t + uu_x = G(u^\tau). \tag{1.2}$$

Equation (1.2) is similar to Hopf or inviscid Burgers' equation [1]. However, (1.2) has a delay term, which makes the equation difficult to be solved [2]. Applications of delay differential equations can be found in [2, 3, 4, 5].

One of the powerful methods for finding analytical solutions of differential equations is *group analysis*. Group analysis was introduced by Shopus Lie in 1895 [6, 7, 8]. Group analysis is applied for finding analytical solutions of many types of ODEs and PDEs [8]. Later, it was developed to apply to integro-differential equations [8], delay differential equations [3], functional differential equations [4, 5] and stochastic differential equations [9].

In this manuscript, group analysis is applied to find symmetries of equation (1.2). Classification of (1.2) with respect to groups of symmetries admitted by the equation is done. Representations of analytical solutions and reduced equations are also presented.

2. Applications of group analysis to delay differential equations

Let $\varphi : \Omega \times \triangle \to \Omega$ be a transformation where Ω is a set of variables (x,t,u) and $\triangle \subset \mathbb{R}$ is a symmetric interval with respect to zero. Variable ε is considered as a parameter of transformation φ, which transforms variable (x,t,u) to $(\bar{x},\bar{t},\bar{u})$ of the same space. Let $\varphi(x,t,u;\varepsilon)$ be denoted by $\varphi_\varepsilon(x,t,u)$. The set of functions φ_ε forms *a one-parameter transformation group of space* Ω if the following properties hold [6, 7, 8]:

(1) $\varphi_0(x,t,u) = (x,t,u)$ for any $(x,t,u) \in \Omega$;
(2) $\varphi_{\varepsilon_1}\left(\varphi_{\varepsilon_2}(x,t,u)\right) = \varphi_{\varepsilon_1+\varepsilon_2}(x,t,u)$ for any $\varepsilon_1, \varepsilon_2, \varepsilon_1+\varepsilon_2 \in \triangle$ and $(x,t,u) \in \Omega$;
(3) if $\varphi_\varepsilon(x,t,u) = (x,t,u)$ for any $(x,t,u) \in \Omega$, then $\varepsilon = 0$.

The other notations $\bar{x} = \varphi^x(x,t,u;\varepsilon)$, $\bar{t} = \varphi^t(x,t,u;\varepsilon)$, $\bar{u} = \varphi^u(x,t,u;\varepsilon)$ are used as the same meaning as $\varphi_\varepsilon(x,t,u) = (\bar{x},\bar{t},\bar{u})$. The transformed variable u with delay term and it's derivatives are defined by $\bar{u}^\tau = \bar{u}(\bar{x},\bar{t}-\tau)$ and $\bar{u}_{\bar{x}} = \partial\bar{u}/\partial\bar{x}, \bar{u}_{\bar{t}} = \partial\bar{u}/\partial\bar{t}$, respectively. Suppose that the transformations map a solution $u(x,t)$ of differential equation

$$F(x,t,u,u^\tau,u_x,u_t) = 0 \tag{2.1}$$

into a solution of the same equation. These transformations are called *symmetries*. In [5], it is shown that for a symmetry

$$\left.\frac{\partial F(\bar{x},\bar{t},\bar{u},\bar{u}^\tau,\bar{u}_{\bar{x}},\bar{u}_{\bar{t}})}{\partial\varepsilon}\right|_{\varepsilon=0,\ (2.1)} = \left.\tilde{X}F(x,t,u,u^\tau,u_x,u_t)\right|_{(2.1)} \equiv 0. \tag{2.2}$$

The operator $\tilde{X}$ is defined by

$$\tilde{X} = (\zeta - u_x\xi - u_t\eta)\partial_u + (\zeta^\tau - u_x^\tau\xi^\tau - u^\tau u_t^\tau\eta)\partial_{u^\tau} + \zeta^{u_x}\partial_{u_x} + \zeta^{u_t}\partial_{u_t},$$

where

$$\xi(x,t,u) = \frac{\partial\varphi^x}{\partial\varepsilon}(x,t,u;0), \quad \eta(x,t,u) = \frac{\partial\varphi^t}{\partial\varepsilon}(x,t,u;0),$$

$$\zeta(x,t,u) = \frac{\partial\varphi^u}{\partial\varepsilon}(x,t,u;0), \quad \xi^\tau = \xi(x,t-r,u^\tau),$$

$$\eta^\tau = \eta(x,t-r,u^\tau), \quad \zeta^\tau = \zeta(x,t-r,u^\tau),$$
$$\zeta^{u_x} = D_x\left(\zeta - u_x\xi - u_t\eta\right), \quad \zeta^{u_t} = D_t\left(\zeta - u_x\xi - u_t\eta\right),$$

$$\begin{aligned} D_x &= \partial_x + u_x\partial_u + u_x^\tau\partial_u^\tau + u_{xx}\partial_{u_x} + u_{xt}\partial_{u_t} + \dots, \\ D_t &= \partial_t + u_t\partial_u + u_t^\tau\partial_u^\tau + u_{xt}\partial_{u_x} + u_{tt}\partial_{u_t} + \dots. \end{aligned}$$

The operator $\tilde{X}$ is called *a canonical Lie–Bäcklund infinitesimal generator* of a symmetry. Equation (2.2) is called *a determining equation.* Lie's theory [6, 7, 8] shows that there is a one-to-one correspondence between the generator and a symmetry. This generator is also equivalent to *an infinitesimal generator* [7]

$$X = \xi\partial_x + \eta\partial_t + \zeta\partial_u. \tag{2.3}$$

By the theory of existence of a solution of a delay differential equation, the initial value problem has a particular solution corresponding to a particular initial value. Because initial values are arbitrary, variables u, u^τ and their derivatives can be considered as arbitrary elements. Since every transformed-solution $\bar{u}(\bar{x},\bar{t})$ is a solution of equation (2.1), the determining equation must be identical to zero. Thus, if determining equation (2.2) is written as a polynomial of variables and their derivatives, the coefficients of these variables in the equations must vanish. In order to solve a determining equation, one solves the several equations of these coefficients. This method is called *splitting the determining equation.* Unknown functions ξ, η and ζ can be obtained from this process.

3. Symmetries of (1.2)

We define determining equation for $u_t + uu_x = G(u^\tau)$ by letting $F = u_t + uu_x - G(u^\tau)$, then

$$\tilde{X}(u_t + uu_x - G(u^\tau))\Big|_{u_t = G - uu_x} \equiv 0. \tag{3.1}$$

Splitting determining equation (3.1) with respect to u_x^τ, u_x and later with respect to u^τ, u, the equation is simplified to

$$\xi_1\left(G'u^\tau - G\right) = 0, \tag{3.2}$$

where the unknown function ξ, η and ζ are

$$\xi = \xi_1 x + \xi_2, \quad \eta = \eta_1, \quad \zeta = \xi_1 u.$$

Here, ξ_1, ξ_2, η_1 are constants.

3.1. Kernel. The set of symmetries, which are admitted for any functional appeared in the equation is called *a kernel* of admitted generators. In this case, $G'u^\tau$ and G are arbitrary. This implies that coefficients of $G'u^\tau$ and G vanish, $\xi_1 = 0$. Unknown functions ξ, η, ζ are

$$\xi = \xi_2, \ \eta = \eta_1, \ \zeta = 0.$$

For the sake of convenience, let arbitrary constants ξ_2, η_1 be denoted by C_1, C_2, respectively. The obtained infinitesimal generator is

$$X = C_1\partial_x + C_2\partial_t. \tag{3.3}$$

This generator is admitted for any functional G. By Lie's theory, symmetry is derived from the infinitesimal generator [7, 8]:

$$\bar{x} = x + C_1\varepsilon, \ \bar{t} = t + C_2\varepsilon, \ \bar{u} = u. \tag{3.4}$$

3.2. Extensions of the kernel. *Extensions* are symmetries for the particular functional G only. In this case, there exists $G(u^\tau)$ satisfying equation (3.2). Here, the extension of kernel (3.3) will be considered. Since $\xi=\xi_2,\ \eta=\eta_1,\ \zeta=0$ are considered in the case of kernel, then functions ξ,η and ζ for this case are

$$\xi=\xi_1 x,\quad \eta=0,\quad \zeta=\xi_1 u.$$

For the nontrivial case, $\xi_1\neq 0$ and a solution of equation (3.2) is

$$G(u^\tau)=ku^\tau,$$

where k is a nonzero arbitrary constant. For the sake of convenience, let ξ_1 be denoted by C_3. The extension of kernel (3.3) is

$$X=C_3\left(x\partial_x+u\partial u\right). \tag{3.5}$$

The symmetry derived from X is

$$\bar{x}=xe^{C_3\varepsilon},\quad \bar{t}=t,\quad \bar{u}=ue^{C_3\varepsilon}. \tag{3.6}$$

4. Representations of solutions

Invariants are functions such that their values do not change by symmetries [6, 7, 8], i.e.

$$\Psi(x,t,u)=\Psi(\bar{x},\bar{t},\bar{u}),$$

where Ψ is an invariant for a symmetry $\varphi_\varepsilon(x,t,u)=(\bar{x},\bar{t},\bar{u})$. If $X=\xi\partial_x+\eta\partial_t+\zeta\partial_u$ is an infinitesimal generator for a symmetry φ_ε, then

$$X\Psi(x,t,u)=0. \tag{4.1}$$

Invariants of symmetries are found by solving differential equation (4.1) [7]. The system of characteristic equations for the infinitesimal generator (2.3) is

$$\frac{dx}{\xi}=\frac{dt}{\eta}=\frac{du}{\zeta}.$$

Representations of solutions are obtained from the invariants.

4.1. Representations of solutions for equation (1.2) with arbitrary functional G. For infinitesimal generator (3.3), the system of characteristic equations is

$$\frac{dx}{C_1}=\frac{dt}{C_2}=\frac{du}{0}.$$

Solving the system of equations, the invariants are u and C_2x-C_1t. For constructing a representation of solution [6, 7], the relation between these two invariants is

$$u=f_1(C_2x-C_1t), \tag{4.2}$$

where f_1 is an arbitrary function. We call u in equation (4.2) *a representation of solution of equation (1.2) for the infinitesimal generator (3.3).*

4.2. Representations of solutions for $G=ku^\tau$. The infinitesimal generator for equation

$$u_t+uu_x=ku^\tau \tag{4.3}$$

is the linear combination of kernel (3.3) and extension (3.5) :

$$X=\left(C_1+C_3x\right)\partial_x+C_2\partial_t+C_3u\partial_u. \tag{4.4}$$

Thus, the system of characteristic equations for infinitesimal generator (4.4) is

$$\frac{dx}{C_1+C_3x}=\frac{dt}{C_2}=\frac{du}{C_3u}.$$

Let $C_2=0$. In this case, the invariants are t and $\frac{u}{x+C_1/C_3}$.

Since C_1 and C_3 are arbitrary and $C_3\neq 0$, for the sake of convenience, we denote $C_4=C_1/C_3$. *The representation of a solution for equation (1.2) with the functional $G=ku^\tau$* is

$$u=(x+C_4)f_2(t), \tag{4.5}$$

where f_2 is an arbitrary function and C_4 is an arbitrary constant.

Let $C_2 \neq 0$. In this case, the invariants are $(x + C_4)\, e^{-(C_3/C_2)t}$ and $ue^{-(C_3/C_2)t}$. *The representation of a solution for equation (1.2) with the functional* $G = ku^{\tau}$ is

$$u = e^{(C_3/C_2)t} f_3 \left((x + C_4)\, e^{-(C_3/C_2)t} \right), \tag{4.6}$$

where f_3 is an arbitrary function. Let $C_5 = C_3/C_2$, equation (4.6) is simply written as

$$u = e^{C_5 t} f_3 \left((x + C_4)\, e^{-C_5 t} \right).$$

5. Reduced equations

Representations of solutions obtained in section 4 simplify equation (1.2). They reduce the number of independent variables appearing in the equation. Substituting the representations into the equation, equation (1.2) is reduced to an ordinary differential equation, which is called *a reduced equation*.

5.1. $u = f_1(C_2 x - C_1 t)$**.** Substituting u into equation (1.2), the equation is transformed to

$$-C_1 f_1'(\theta) + C_2 f_1(\theta) f_1'(\theta) = G(f_1(\theta + C_1 \tau)),$$

where $\theta = C_2 x - C_1 t$. This equation may be written in the other form,

$$f_1'(\theta) = \frac{G(f_1(\theta + C_1 \tau))}{C_2 f_1(\theta) - C_1}. \tag{5.1}$$

5.2. $u = (x + C_4) f_2(t)$**.** Substituting u into equation (4.3), the equation is transformed to

$$(x + C_4) f_2'(t) + (x + C_4) \left[f_2(t) \right]^2 = k(x + C_4) f_2(t - \tau).$$

It can be simplified to

$$f_2'(t) = k f_2(t - \tau) - \left[f_2(t) \right]^2. \tag{5.2}$$

5.3. $u = e^{C_5 t} f_3 \left((x + C_4)\, e^{-C_5 t} \right)$**.** Substitute u into equation (4.3), the equation is transform to

$$C_5 f_3(\phi) - C_5 \phi f_3'(\phi) + f_3(\phi) f_3'(\phi) = k e^{-C_5 \tau} f_3 \left(e^{C_5 \tau} \phi \right),$$

where $\phi = (x + C_4)\, e^{-C_5 t}$. The other form of the equation is

$$f_3'(\phi) = \frac{C_5 f_3(\phi) - k e^{-C_5 \tau} f_3 \left(e^{C_5 \tau} \phi \right)}{f_3(\phi) - C_5 \phi}. \tag{5.3}$$

Note that equation (5.1), (5.2) and (5.3) are not typical ODEs, they are functional ODEs [5].

6. Conclusion

Symmetries, representation of solutions of equation (1.2) and reduced equations are presented in section 3, 4 and 5, respectively. Equation (1.2) is classified with respect to the symmetries into the case of $G(u^{\tau}) = ku^{\tau}$ (symmetry is (3.6)) and otherwise (symmetry is (3.4)). By the review literature, there are not many examples of applications of group analysis to delay differential equations. This manuscript presents another example.

Acknowledgements This work is supported by Commission On Higher Education and Thailand Research Fund, grant number MRG4880145, and Suranaree University of Technology, PED project. Author is deeply indebted to Prof.Sergey Meleshko for his numerous help. I would like to express my sincere thank to him and Asst.Prof.Dr.Apichai Hematulin.

References

[1] J. D. Logan, *An Introduction to Nonlinear Partial Differential Equations,* John Wiley & Sons, New York, 1999.

[2] R. D. Driver, *Ordinary and Delay Differential Equations,* Springer-Verlag, New York, 1977.

[3] J. Tanthanuch and S. V. Meleshko, On definition of an admitted Lie group for functional differential equations, *Communication in Nonlinear Science and Numerical Simulation,* **9** (2004), 117–125.

[4] J. Tanthanuch and S. V. Meleshko, Application of group analysis to delay differential equations, *Nonlinear acoustics at the beginning of the 21st century,* Moscow State University, Moscow, 2002, pp. 607–610.

[5] J. Tanthanuch, *Application of Group Analysis to Functional Differential Equations,* Ph.D. Thesis, Nakhonratchasrima, Suranaree University of Technology, Thailand, 2003.

[6] L. V. Ovsiannikov, *Group Analysis of Differential Equations,* Academic Press, New York, 1982.

[7] N. H. Ibragimov, *Elementrary Lie Group Analysis and Ordinary Differential Equations,* John Wiley & Sons, London, 1999.

[8] N. H. Ibragimov, *Lie group analysis of differential equations,* Vol.1-3., CRC Press, Florida, 1994.

[9] B. Srihirun, S. V. Meleshko and E. Schulz, On definition of an admitted Lie group for stochastic differential equations, *Communication in Nonlinear Science and Numerical Simulation,* in press. (available online at `www.sciencedirect.com`)