

## Application of the Fibonacci Numbers to Mathematical Problems

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### Abstract

The Fibonacci numbers are numbers of the integer sequence 1, 1, 2, 3, 5, 8, 13, ..., which are defined by  $a_1 = a_2 = 1$  and  $a_{n+2} = a_{n+1} + a_n$ , for  $n = 1, 2, 3, \dots$ . By the history, the sequence of these numbers was first introduced officially by Leonardo Pisano Bigollo for solving the mathematical problem involving the growth of population of rabbits based on idealized assumptions. The objective of this research was to consider the other three mathematical problems which could be expressed by the Fibonacci numbers also. The problems were 1) determine the number of patterns for bricks rearrangement, 2) determine the number of subsets of  $\{1, 2, \dots, n\}$  which do not contain two consecutive numbers, and 3) determine the number of paths for travelling between 5 cities such that the traveler can change direction at any intermediate city. The recurrence relation concept was used to show the connection between the problems and Fibonacci numbers. It showed that high school mathematics was able to explain these mathematical problems. In order to ease the explanation, animations and slides by Microsoft PowerPoint were prepared.

**Keywords:** Fibonacci numbers, mathematics, recurrence relation

### Introduction/Problem

In the present, people face the difficulty and try to overcome it all the times. Hence it is necessary to improve the skill of solving problem. Mathematics is a subject that provides logical thinking. It is used to prove the thought reasonably. Since many problems in mathematics are simplified from the problems in daily life, the study of mathematics helps to figure out the solutions of both math-problems and real life problems. Moreover mathematics is always applied to give explanations for science phenomena. It is also believed that mathematics is essential to innovation and technology (Pipitkul, 2003 and Tipkong, 2002). However, Thai students did not perform well in the mathematics placement test, ordinary national education test (O-NET) of Thailand from B.E. 2551-2553. The average score was low and tended to lower (NIETS, 2011). Students usually think that mathematics is not necessary for their real life. Hence mathematics in natural problem may inspire learners to feel that theory and practice can be jointed with each other.

“Sequences” is a content studied in high school, which is used for the higher level study, e.g. calculus. It concerns about the pattern of numbers. The subtopics in this content provide skills of observation, pattern recognition and calculation by using formulas. By the history, there are many well known sequences explaining the natural processes, for example, arithmetic sequence, geometric sequence and Cauchy sequence. One of very famous sequences is Fibonacci sequence. The Fibonacci numbers are numbers of the integer sequence 1, 1, 2, 3, 5, 8, 13, ..., which are defined by  $a_1 = a_2 = 1$  and  $a_{n+2} = a_{n+1} + a_n$ , for  $n = 1, 2, 3, \dots$ . The sequence of

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these numbers was first introduced officially by Leonardo Pisano Bigollo, who was known as Fibonacci. The sequence was used for solving the mathematical problem involving the growth of population of rabbits based on idealized assumptions. Assuming that: a newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year? (Wikipedia, online) This is an example of mathematics in natural problem.

The objective of this research was to consider the other three mathematical problems which could be expressed by the Fibonacci numbers also. The problems were 1) determine the number of patterns for bricks rearrangement, 2) determine the number of subsets of  $\{1,2,\dots,n\}$  which do not contain two consecutive numbers, and 3) determine the number of paths for travelling between 5 cities such that the traveler can change direction at any intermediate city. The recurrence relation concept was used to show the connection between the problems and Fibonacci numbers.

### Design/Procedure

Solutions of the considered problems are showed as follows.

#### 1) Determine the number of patterns for bricks rearrangement

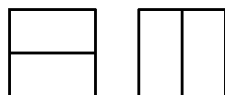
Suppose that there is a  $2 \times n$  hole on the wall, and we wish to fill it using  $1 \times 2$  boards. A board may be placed in either orientation. What is the number of different ways of filling in this hole?

#### Example

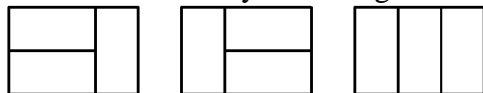
There is only one way of filling  $2 \times 1$  hole.



There are two ways of filling  $2 \times 2$  hole.



There are three ways of filling  $2 \times 3$  hole.



To solve this problem, we start on letting the number of ways be  $a_n$ . We have  $a_0 = 1$  vacuously, and from the diagram above,  $a_1 = 1$ ,  $a_2 = 2$  and  $a_3 = 3$ . For  $n \geq 2$ , we classify a filling as type I if two horizontal boards are touching the right edge of the hole, and as type II if one vertical board is touching the right edge. Each filling of type I can be obtained from a filling of the  $2 \times (n-2)$  hole by adding two horizontal boards. Hence the number of type I filling is  $a_{n-2}$ . Similarly, the number of type II filling is  $a_{n-1}$ , and we have  $a_n = a_{n-1} + a_{n-2}$ . Thus the solution of this problem is a sequence of Fibonacci numbers likewise,

$$a_0 = a_1 = 1, a_2 = a_1 + a_0 = 2, a_3 = a_2 + a_1 = 3, a_4 = a_3 + a_2 = 5 \text{ and} \\ a_n = a_{n-1} + a_{n-2} \text{ for } n=2,3,4,\dots$$

**2) Determine the number of subsets of  $\{1,2,\dots,n\}$  which do not contain two consecutive numbers.**

**Example.** Determine the number of subsets of  $\{1, 2, 3, 4, 5, 6, 7\}$  of size 3 which do not contain two consecutive numbers.

The size of this problem is small enough for us to work out the desirable subsets, of which there are 10. They are listed in the chart below on the left.

135	123
136	124
137	125
146	134
147	135
157	145
246	234
247	235
257	245
357	345

Clearly, we cannot do this when the size of the problem is significantly larger. We use a smart transformation. In the chart above, the first digits of the numbers on the right are the same as the corresponding ones on the left. The second digits are 1 less and the third digits are 2 less. In the numbers on the left, the digits differ by at least 2 from column to column. Thus in the numbers on the right, the three digits are different. Since the last digit on the left is 7, the largest digit on the right is 5. It follows that the numbers on the right represent subsets of  $\{1, 2, 3, 4, 5\}$ . Since this transformation is reversible, the answer to this is just the number of all subsets of size 3 of  $\{1, 2, 3, 4, 5\}$ , which is  $nCr(5,3)=5!/(2!3!)=10$ . By the given idea, we found that numbers of subsets of  $\{1, 2, 3, 4, 5, 6, 7\}$  of different sizes are in the following chart.

size	number
0	$nCr(8,0)=1$
1	$nCr(7,1)=7$
2	$nCr(6,2)=15$
3	$nCr(5,3)=10$
4	$nCr(4,4)=1$
total	34

In general, let  $S_n$  be a set of all subsets of  $\{1,2,\dots,n\}$  which do not contain two consecutive numbers. We have  $S_0 = \{\emptyset\}$  where  $|S_0| = 1$ ,  $S_1 = \{\emptyset, \{1\}\}$  where  $|S_1| = 2$ ,  $S_2 = \{\emptyset, \{1\}, \{2\}\}$  where  $|S_2| = 3$  and  $S_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}\}$  where  $|S_3| = 5$ . For  $n \geq 2$ , we classify an element  $S_n$  of as type I if it is a subset of  $\{1,2,\dots,n\}$  containing  $n$ , and as type II if the subset contains no  $n$ . Because we do not want the consecutive number, the type I subset can be obtained from a union of an element of  $S_{n-2}$  with  $\{n\}$ , where the type II subset can be obtained directly from an element of  $S_{n-1}$ . For example, type I elements of  $S_3$  are  $\{3\}$  and  $\{1,3\}$  where they are produced from the union of elements of  $S_1$ ,  $\emptyset$  and  $\{1\}$ , with  $\{3\}$ . Type II elements of  $S_3$  are  $\emptyset, \{1\}$  and  $\{2\}$ , which are elements of  $S_2$ . Thus number of elements of  $S_n$  is equal to  $|S_{n-2}|+|S_{n-1}|$ , i.e.

$$|S_n|=|S_{n-2}|+|S_{n-1}|, \text{ for } n=2,3,4,\dots$$

It shows that the solution of this problem has the same structure with a sequence of Fibonacci numbers.

**3) Determine the number of paths for travelling between 5 cities such that the traveller can change direction at any intermediate city.**

**Example.** Along the Grand Trunk Road were the cities *Lahore*, *Umballa*, *Delhi*, *Alighur* and *Benares* in that order. Kim started from *Lahore* and headed for *Benares*. At any intermediate city, he would decide either to continue in the same direction or to turn back. His journey would come to an end if he reached *Benares* or returned to *Lahore*. If he changed directions three times altogether, his path could have taken any of the following forms:

- (a) *Lahore — Umballa — Delhi — Umballa — Delhi — Umballa — Lahore.*
- (b) *Lahore — Umballa — Delhi — Alighur — Delhi — Umballa — Delhi — Umballa — Lahore.*
- (c) *Lahore — Umballa — Delhi — Alighur — Delhi — Alighur — Delhi — Umballa — Lahore.*
- (d) *Lahore — Umballa — Delhi — Umballa — Delhi — Alighur — Delhi — Umballa — Lahore.*
- (e) *Lahore — Umballa — Delhi — Alighur — Delhi — Umballa — Delhi — Alighur — Delhi — Umballa — Lahore.*

Let  $K_n$  denote the number of possible forms for Kim's path if he changed directions  $n$  times. Then  $K_0 = 1$  since he would go directly to *Benares*, his path taking the unique form

- (f) *Lahore — Umballa — Delhi — Alighur — Benares.*

We have  $K_1 = 2$  since he could turn back to *Lahore* from *Delhi* or *Alighur*, his path taking the possible forms

- (g) *Lahore — Umballa — Delhi — Umballa — Lahore.*
- (h) *Lahore — Umballa — Delhi — Alighur — Delhi — Umballa — Lahore.*

Consider  $K_2$ . He would end up in *Benares*, and the last change in directions must occur in *Umballa* or *Delhi*. If it occurred in *Delhi*, this means that he must have returned from *Alighur*, and had gone there earlier from *Delhi*. If we shorten the last segment "*Delhi — Alighur — Delhi*" to just "*Delhi*", we will get the unique form for  $K_0$ . On the other hand, if the last change in directions occurred in *Umballa*, this means that he could have returned to *Lahore* had he not done so. If we replace the segment "*Delhi — Alighur — Benares*" at the very end by "*Lahore*", we will get either of the forms for  $K_1$ . This means that  $K_2 = K_0 + K_1 = 3$ , and the three possible forms of the path are

- (i) *Lahore — Umballa — Delhi — Alighur — Delhi — Alighur — Benares.*
- (j) *Lahore — Umballa — Delhi — Umballa — Delhi — Alighur — Benares.*
- (k) *Lahore — Umballa — Delhi — Alighur — Delhi — Umballa — Delhi — Alighur — Benares.*

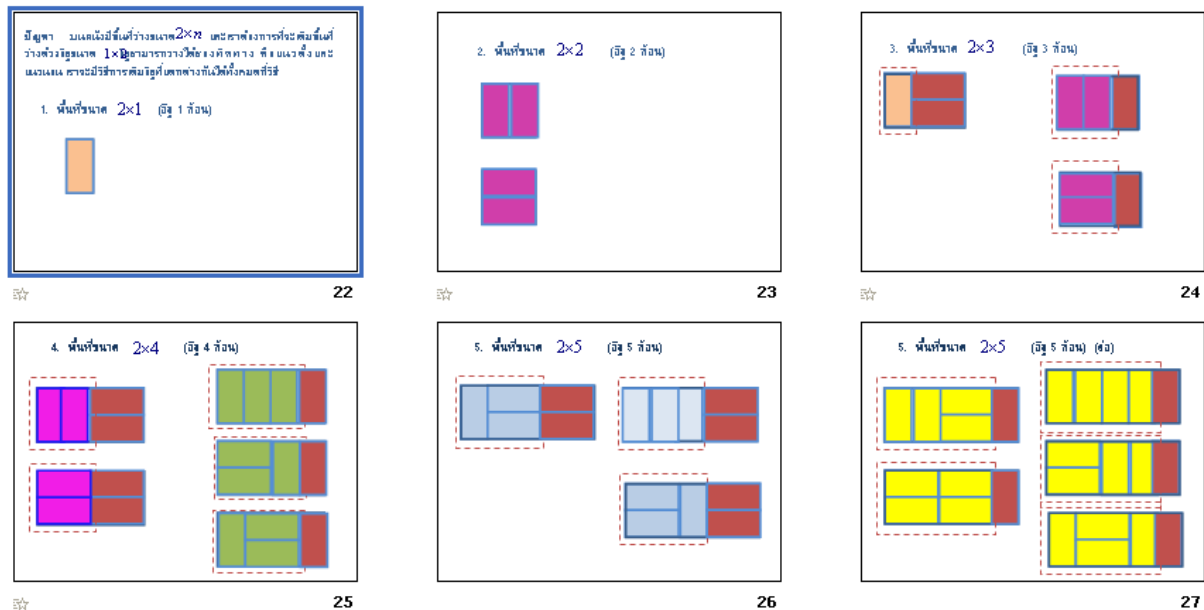
Note that (i) arises from (f), (j) from (g) and (k) from (h).

We claim that  $K_3 = K_1 + K_2 = 5$ . This time, Kim would be back in *Lahore*. Hence the last change in directions must occur in *Delhi* or *Alighur*. If it occurred in *Delhi*, this means that he must have returned from *Umballa*, and had gone there earlier from *Delhi*. If we shorten the last segment "*Delhi — Umballa — Delhi*" to just "*Delhi*", we will get either of the forms for  $K_1$ . Note that (a) arises from (g) and (b) from (h). On the other hand, if the last change in directions occurred in *Alighur*, this means that he could have not gone onto *Benares*. If we

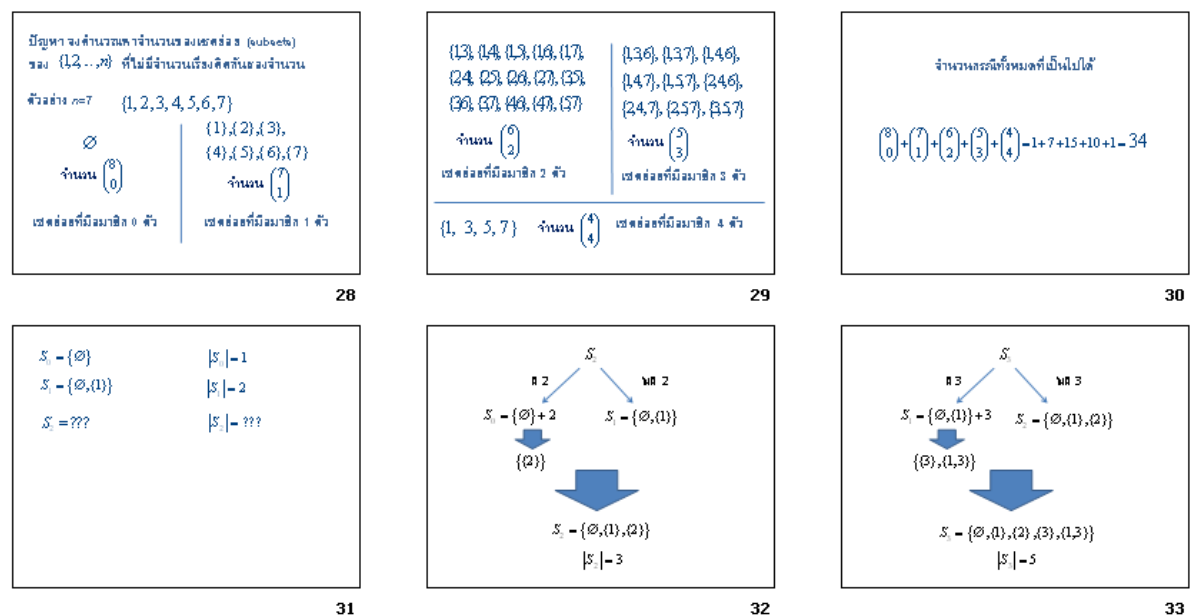
replace the segment “Delhi — Umballa — Lahore” at the very end by “Benares”, we will get one of the forms for  $K_2$ . Note that (c) arises from (i), (d) from (j) and (e) from (k). This justifies our claim. In the same way, we have  $K_4 = K_2 + K_3 = 8$ ,  $K_5 = K_3 + K_4 = 13$ ,  $K_6 = 21$ ,  $K_7 = 34$ ,  $K_8 = 55$ ,  $K_9 = 89$  and  $K_{10} = 144$ .

### Finding/Analysis

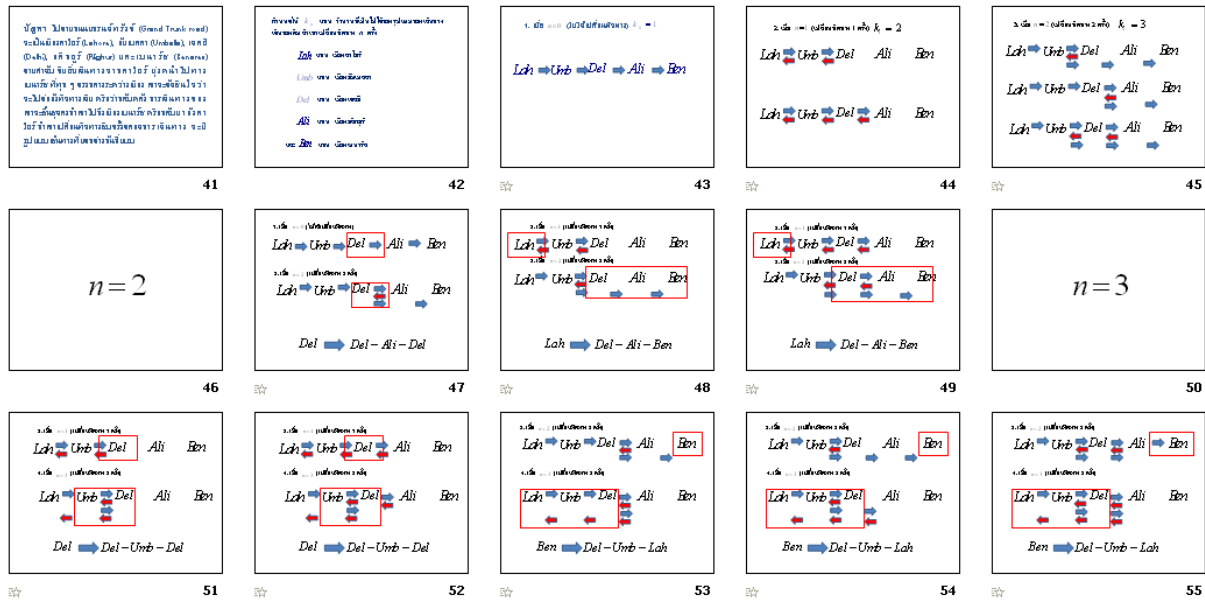
Even we could figure out the solution of the problems via the sequence of Fibonacci numbers, but it is hard to weak background students to understand the explanation. Then the slides and animations by Microsoft Office Powerpoint were prepared. The examples of slides are showed as follows.



**Figure 1** Some slides for problem “Determine the number of patterns for bricks rearrangement” explanation



**Figure 2** Some slides for problem “Determine the number of subsets of  $\{1,2,\dots,n\}$  which do not contain two consecutive numbers” explanation



**Figure 3** Some slides for problem “Determine the number of paths for travelling between 5 cities such that the traveller can change direction at any intermediate city” explanation

### Recommendation

It was found that the suggested problems could be explained by the sequence of Fibonacci numbers. The analysis of the problem is helpful to the reader for thinking in conceptual. This is an example of application of high school mathematics to a problem.

### Acknowledgement

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