103625 APPLIED ANALYSIS

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Chapter 1

Groups, Rings and Fields

1.1 Binary operation

Let S be a set. A mapping $\circ$,

$$\circ : S \times S \mapsto S,$$

is called a binary operation (of S into itself). If x, y are elements of S, the image of the pair (x, y) under the operation $\circ$ is sometimes called the product under the binary operation.

Remark

1. $\circ(a, b)$ has to be defined for any ordered pair $(a, b) \in S \times S$ and $\circ(a, b)$ is well defined.

2. $\circ(a, b) \in S$ for all $a, b \in S$.

   For this property, we say that S is closed under $\circ$. For the sake of simplicity, we may denote $\circ(a, b)$ by $a \circ b$ or sometimes ab.

Example 1.1. Let $\mathfrak{F} = \{f | f : \mathbb{R} \mapsto \mathbb{R}\}$. Define $\circ$ on $\mathfrak{F}$ by $f \circ g = \circ(f, g) = h$ where $h(x) = f(x) + g(x)$ for any $f, g \in \mathfrak{F}$ and $x \in \mathbb{R}$. It is obvious that $\circ$ is a binary operation on $\mathfrak{F}$.

Example 1.2. Let $\circ$ be defined on a set of integer numbers $\mathbb{Z}$ by $a \circ b = \dfrac{a}{b}$. We can see that $\circ$ is not closed under $\mathbb{Z}$.

Example 1.3. Let $\circ$ be defined on a set of rational numbers $\mathbb{Q}$ (or a quotient set $\mathbb{Q}$) by $a \circ b = \dfrac{a}{b}$.

Question : Is "$\circ$" a binary operation on $\mathbb{Q}$?

Answer : No, it is not. Since "$\circ$" is not defined for (3,0).

Remark (3,0) is just a counter example. There are many examples to show that $\circ$ is not defined for some ordered pairs, i.e. (0,0), (0.5,0), (1,0), (-2,0), ...

Example 1.4. Let $\circ$ be defined on a quotient set $\mathbb{Q}$ by $a \circ b = c$, where c is an rational number which is greater than both a and b. Since there are many elements which are greater than a and b[1], the definition gives an ambiguous result. $\circ$ is *not well defined*.

Definition 1.1. Let $\circ$ be a binary operation on a set S.

- $\circ$ has an *associative* property if

$$(a \circ b) \circ c = a \circ (b \circ c), \qquad \forall\ a, b, c \in S.$$

- $\circ$ has a *commutative* property if

$$a \circ b = b \circ a \qquad \forall\ a, b \in S.$$

Example 1.5. Consider the additive operation "+" and multiplicative operation "$\cdot$". They are binary operations on the set of real numbers $\mathbb{R}$ which have associative and commutative properties, i.e.

$$a + (b + c) = (a + b) + c \qquad \text{and} \qquad a + b = b + a,$$

$$\text{also} \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{and} \qquad a \cdot b = b \cdot a \qquad \forall a, b, c \in \mathbb{R}.$$

[1]Note that a set of rational numbers $\mathbb{Q}$ is not bounded above, i.e. we cannot find an element $x \in \mathbb{Q}$ such that $a \leq x$ for all $a \in \mathbb{Q}$.

Example 1.6. Define $\min(a,b) = \begin{cases} a & \text{if } a \leq b, \\ b & \text{if } a > b \end{cases}$. Let $\odot$ be defined on a set of positive integers $\mathbb{Z}^+$ by

$$a \odot b = \min(a,b) + 2, \qquad \forall\, a, b \in \mathbb{Z}^+.$$

It is easy to see that $\odot$ is a binary operation on $\mathbb{Z}^+$. Further more, $\odot$ has a commutative property but has no property of associativity, i.e

- $a \odot b = \min(a,b) + 2$ and $b \odot a = \min(b,a) + 2$.

  Thus $a \odot b = b \odot a$ for all $a, b \in \mathbb{Z}^+$.

- However

$$\begin{aligned} (1 \odot 2) \odot 3 &= [\min(1,2) + 2] \odot 3 \\ &= (1+2) \odot 3 \\ &= 3 \odot 3 \\ &= \min(3,3) + 2 \\ &= 3 + 2 \;=\; 5 \end{aligned}$$

and

$$\begin{aligned} 1 \odot (2 \odot 3) &= 1 \odot [\min(2,3) + 2] \\ &= 1 \odot (2+2) \\ &= 1 \odot 4 \\ &= \min(1,4) + 2 \\ &= 1 + 2 \;=\; 3. \end{aligned}$$

  This shows that $(a \odot b) \odot c \neq a \odot (b \odot c)$ for some $a, b, c \in \mathbb{Z}^+$.

Example 1.7. Let $\triangle$ be defined on $\mathbb{Z}^+$ by $a \triangle b = b$ for all $a, b \in \mathbb{Z}^+$. $\triangle$ is a binary operation with associative property :

for $a, b, c \in \mathbb{Z}^+$

$$\begin{aligned} a \triangle (b \triangle c) &= a \triangle c = c \\ (a \triangle b) \triangle c &= b \triangle c = c \end{aligned}$$

So $a\triangle(b\triangle c) = (a\triangle b)\triangle c$ for $a, b, c \in \mathbb{Z}^+$. However, $\triangle$ is not commute :

$$3\triangle 11 = 11 \qquad \text{but} \qquad 11\triangle 3 = 3.$$

Example 1.8. Let $\star$ be defined on a set of integers $\mathbb{Z}$ by $a \star b = a^2 - b^2$ for all $a, b \in \mathbb{Z}$. $\star$ is a binary operation which has no either associative and commutative properties :

$$\begin{aligned} 4 \star (3 \star 2) &= 4 \star (3^2 - 2^2) = 4 \star 5 \\ &= 4^2 - 5^2 = -9 \\ \text{while} \qquad (4 \star 3) \star 2 &= (4^2 - 3^2) \star 2 = 7 \star 2 \\ &= 7^2 - 2^2 = 45 \end{aligned}$$

also

$$1 \star 2 = 1^2 - 2^2 = -3 \neq 3 = 2^2 - 1^1 = 2 \star 1$$

EXERCISE

Consider whether the following operations on the given sets are binary operations or not. If they are determine whether they have properties of associative and commutative or not.

1. $a \circ b = a - b$ on $\mathbb{Z}^+$
2. $a \circ b = a - b$ on $\mathbb{Z}$
3. $a \circ b = 2^{ab}$ on $\mathbb{Z}^+$
4. $a \circ b = \sqrt{|ab|}$ on $\mathbb{Q}$
5. $a \circ b = a \ln b$ on a set of positive real numbers $\mathbb{R}^+$
6. $a \circ b = a + b$ on $S = \{-3, -2, -1, 0, 1, 2, 3\}$

1.2 Groups and Elementary Properties

Definition 1.2. $(G, \circ)$ is a *group* if G is not an empty set and $\circ$ is *a binary operation on* G which has the following properties :

1. $\circ$ has an associative property, i.e.

$$a \circ (b \circ c) = (a \circ b) \circ c, \qquad \forall a, b, c \in G.$$

2. There exists $e \in G$ such that

$$a \circ e = e \circ a = a, \qquad \forall a \in G.$$

   We call "e" *an identity element* of G under $\circ$.

3. For any element $a \in G$, there must be $b \in G$ such that

$$a \circ b = b \circ a = e.$$

   We call "b" *an inverse element of* a. For the sake of convenience, we always denote an inverse element of a by a^{-1}.

Remark

- Group G is composed by 2 important parts which are set G and binary operation $\circ$. For the official notation, we denote group by $(G, \circ)$. However, for the sake of convenience, "G is a group." will be briefly denoted in stead of "$(G, \circ)$ is a group." Also ab will be used to mean $a \circ b$. Anyways $(G, \circ)$ may be sometimes used to emphasize the use of the operation $\circ$ on G.

- Some textbooks may say that a group must satisfy four properties, i.e. the three previous properties including "G is closed under the binary operation $\circ$". Actually, our context has considered that as a property of the binary operation.

- By the second property of group, we can say that group must consist of an identity. Thus we find that the smallest group is a set with has only one member $\{e\}$, where the binary operation is defined by $e \circ e = e$. In this case, we can see that e is an inverse of itself. Then set $\{e\}$ satisfies all properties of group.

- The inverse of an identity is itself.

Moreover, $\circ$ may not satisfy all three group properties. If $\circ$ is just a binary operation on G, we call $(G, \circ)$ *a groupoid*. In the case that $\circ$ satisfies only the first property (associative property), we call $(G, \circ)$ *a semigroup*. Also if $\circ$ satisfies the first and second (identity) property, we call $(G, \circ)$ *a monoid*.

Example 1.9. The following groups are often used :

- $(\mathbb{Z}, +)$

  "0" is an identity of this group. For $a \in \mathbb{Z}$, its inverse is $-a$.

- $(\mathbb{Z}_n, +)$

  $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$ is a set of remainders obtained by the division of integers by n, for $n = 2, 3, \ldots$. The identity of this group is $\bar{0}$. The inverse of $\bar{a}$ is $\overline{n-a}$.

  - $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$

    The inverse of $\bar{1}$ is $\bar{1}$, i.e. $\bar{1} + \bar{1} = \overline{1+1} = \bar{2} = \bar{0}$.

  - $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

    The inverse of $\bar{1}$ is $\bar{2}$ and vice versa, i.e. $\bar{1} + \bar{2} = \overline{1+2} = \bar{3} = \bar{0}$.

  - $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

    The inverse of $\bar{1}$ is $\bar{3}$ and vice versa. The inverse of $\bar{2}$ is itself.

- $(\mathbb{U}_n, \cdot)$

  $\mathbb{U}_n = \{\bar{i}\}$ is a set of remainders obtained by the division of positive integers by n and those remainders are relatively prime to n (i is relatively prime to n means GCD of i and n is equal to 1). The identity of this group is $\bar{1}$. The inverse of $\bar{a}$ is $\bar{b}$ such that $\overline{ab} = \bar{1}$.

- $(\mathbb{Q}^*, \cdot)$ where $\mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$ and $\cdot$ is a multiplicative operation.
  "1" is an identity of this group. For $a \in \mathbb{Q}^*$, its inverse is $\dfrac{1}{a}$.

- $(\mathbb{Q}, +)$

- $(\mathbb{R}, +)$

- $(\mathbb{R}^*, \cdot)$ where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\cdot$ is a multiplicative operation.

- $(\mathfrak{F}, \circ)$ where $\mathfrak{F} = \{f | f : S \xrightarrow[\text{onto}]{1-1} S\}$ and $\circ$ is a composition operation.

- $(\mathcal{M}_{m\times n}, +)$ where $\mathcal{M}_{m\times n}$ is a set of $m \times n$ matrices.

  The identity of this group is a zero $m \times n$ matrix $0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. For

  $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \text{its inverse is } -A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix}$$

- $(\mathcal{M}_{n\times n}, \times)$ where $\mathcal{M}_{n\times n}$ is a set of nonsingular $n \times n$ matrices.

  The identity of this group is an identity $n \times n$ matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.

The inverse of matrix A is $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$, where c_{ij} is a cofactor.

Definition 1.3. Group $(G, \circ)$ is *an abelean group* (or *a commutative group*) if

$$ab = ba \qquad \forall a, b \in G.$$

Theorem 1.1. *Let G be a group. G must satisfy the following properties :*

1. *G has a unique identity. (Exercise 3.)*

2. *For each $a \in G$, a has a unique inverse. (Exercise 4.)*

Theorem 1.2 (The rule of cancellation)**.** *Let G be a group. Any $a, b, c \in G$ must satisfy the following properties :*

1. *If $ab = ac$ then $b = c$ (Left cancelation.). (Exercise 7a.)*

2. *If $ba = ca$ then $b = c$. (Right cancelation.) (Exercise 7b.)*

EXERCISE

1. Let $\mathfrak{F} = \{f | f : S \xrightarrow[\text{onto}]{1-1} S\}$. Show that $(\mathfrak{F}, \circ)$ is a group where $\circ$ is a composition operation.

2. Let G be a group and $a \in G$. Show that if $a^2 = a \circ a = a$, it implies that a is an identity.

3. Show that if G is a group. G has only one identity.

4. Let G be a group. Show that for any $a \in G$, a has only one inverse.

5. Let G be a group and $a, b \in G$. Apply exercise 4. to show that $(ab)^{-1} = b^{-1}a^{-1}$

6. Let G be a group and $a \in G$. Show that $(a^{-1})^{-1} = a$

7. Let G be a group and $a, b, c \in G$. Show that

   (a) If $ab = ac$ then $b = c$. (Left cancelation.)

   (b) If $ba = ca$ then $b = c$. (Right cancelation.)

8. Let G be a group and $a, b \in G$. Show that

   (a) There exists a unique $x \in G$ such that $ax = b$

   (b) There exists a unique $y \in G$ such that $ya = b$

9. Let G be a group such that any element in G is an inverse of itself. Show that G is an abelean group.

10. Let G be a group such that $(ab)^2 = a^2b^2$ for $a, b \in G$. Show that G is abelean.

1.3 Symmetric group

Definition 1.4. Let G be a group. If G has n elements, we say that G is *a finite group*. We also say that n is a size or *an order* of group G. If G is an infinite set, we say that G is *an infinite group*.

We always denote an order of a group G by $|G|$.

Let S_n be a set of all permutations σ which are defined by

$$\begin{aligned} \sigma_l(1) &= i_1 \\ \sigma_l(2) &= i_2 \\ &\vdots \\ \sigma_l(n) &= i_n \end{aligned}$$

where $i_j \in \{1, 2, \ldots, n\}$, $j = 1, 2, \ldots, n$, $i_j \neq i_k$ if $j \neq k$ and $l = 1, \ldots, n!$[2] On the other hand, we can say that $\{i_1, i_2, \ldots, i_n\}$ is a rearrangement of numbers $\{1, 2, \ldots, n\}$, i.e. σ is a bijective function from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. For the convenience, we may denote σ in the form

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

Example 1.10. Consider set S_3. All elements in S_3 are

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Theorem 1.3. $(S_n, \circ)$ *is a group under the composition operation* $\circ$.

Proof. The proof of this theorem is left as an exercise.

For the sake of convenient we always denote the permutations in S_n with cycle notations as the followings :

If $i_1, i_2, \ldots, i_r$ are distinct numbers such that $1 \leq i_j \leq n$. We denote $(i_1 i_2 \ldots i_r)$

[2] $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, for $n = 1, 2, 3, \ldots$ and $0! = 1$

instead of σ_k in S_n by

$$\begin{aligned}
\sigma_l(i_1) &= i_2 \\
\sigma_l(i_2) &= i_3 \\
&\vdots \\
\sigma_l(i_{r-1}) &= i_r \\
\sigma_l(i_r) &= i_1
\end{aligned}$$

For numbers which do not appear in the cycle, it means that function σ_l maps that number to itself, i.e. $\sigma_l(j) = j$, for all $j \neq i_1, j \neq i_2, \ldots, j \neq i_n$.

Example 1.11. By example 1.10, we may denote all elements in S_3 in cycle notations as follows :

$$\begin{aligned}
\sigma_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1) = (2) = (3), \\
\sigma_2 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123) = (231) = (312), \\
\sigma_3 &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132) = (213) = (321), \\
\sigma_4 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23) = (32), \\
\sigma_5 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12) = (21), \\
\sigma_6 &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13) = (31).
\end{aligned}$$

Group S_n plays a major role in consideration a finite group. The theorem concerning about this statement will be presented later.

Exercise

1. Prove theorem 1.3

2. Calculate the order of group S_n.

3. Show all elements in S_4 in both classical and cycle form.

1.4 Subgroups

By example 1.9, we can see that $\mathbb{Z} \subseteq \mathbb{Q}$ and both $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are groups. Furthermore, while $\mathbb{Q} \subseteq \mathbb{R}$ and $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ are groups. However, $\mathbb{Q}^+ \subseteq \mathbb{Q}$, $(\mathbb{Q}, +)$ is a group but $(\mathbb{Q}^+, +)$ is not a group. On the other hand, while $\mathbb{Q}^* \subseteq \mathbb{Q}$, $(\mathbb{Q}^*, \cdot)$ is a group but $(\mathbb{Q}, \cdot)$ is not a group. By these examples, we can develop the concept of a group for the smaller set by the following :

Definition 1.5 (Induced Operation)**.** Let G be a group and let H be a subset of G. If for every $a, b \in H$ it is true that the product ab computed in G is also in H, then H is **closed** under the group operation of G. The binary operation on H thus defined is the **induced operation on** H **from** G.

Definition 1.6. Let H be **a subset** of G where $(G, \circ)$ is a group. If H is closed under the **induced** operation $\circ$ and $(H, \circ)$ is a group, we say that H **is a subgroup of** G. We may denote $H \leq G$ or $G \geq H$ to mean H is a subgroup of G.

Notations $H < G$ or $G > H$ may be also used to emphasis that $H \leq G$ but $H \neq G$.

Example 1.12. The well known groups and subgroups in the arithmetic are

$$(\mathbb{Z}, +) \leq (\mathbb{Q}, +) \leq (\mathbb{R}, +) \leq (\mathbb{C}, +),$$

where $\mathbb{C}$ is a set of complex numbers. Other examples are

$$\begin{aligned} (\mathbb{Q}^+,\cdot) &\le (\mathbb{R}^+,\cdot),\\ (\mathbb{Q}^*,\cdot) &\le (\mathbb{R}^*,\cdot)\\ \text{and} \qquad (\mathfrak{C}[0,1],\circ) &\le (\mathfrak{F}[0,1],\circ), \end{aligned}$$

where $\mathfrak{F} = \{f | f : [0,1] \xrightarrow[\text{onto}]{1-1} [0,1]\}$[3], $\mathfrak{C} = \{f | f : [0,1] \xrightarrow[\text{onto}]{1-1} [0,1]$ and f is continuous on $[0,1]\}$[4] and $\circ$ is a composition operation.

Example 1.13. By example 1.11, $S_3 = \{(1),(123),(132),(12),(13),(23)\}$. Let $H = \{(1),(123),(132)\}$. We find that $(H,\circ)$ is a group, i.e. $H \le G$. (The proof that $(H,\circ)$ is a group will be left as an exercise.)

Remark Every group G has at least two subgroups which are $\{e\}$ and G itself, where e is the identity of that group under the same operation. Another subgroup of G that is not G is called a **proper subgroup** of G. G itself is the **improper subgroup** of G. Both $\{e\}$ and G are the **trivial subgroups** of G. All other subgroups are **nontrivial**.

Let G be a group and H be a subgroup of G. Since the binary operation on G takes action to every element in G, the induced operation also does to every element in H. Hence properties of the operation on G translate to subgroup H automatically. By this reason, we can conclude that H is a subgroup of G if and only if H has the following properties.

1. H is closed under the same operation on G, i.e. $a, b \in H$ implies $a \circ b \in H$.

2. The identity e of a group G must be in H.

[3] $[0,1]$ is a set of real numbers x such that $0 \le x \le 1$.

[4] Both definitions and concepts of closed interval $[0,1]$ and "continuous" will be considered precisely in the latter context.

3. If $a \in H$ then $a^{-1} \in H$.

By all these properties, we can conclude as a theorem by the following.

Theorem 1.4. *Let G be a group and $H \subseteq G$ such that $H \neq \phi$. H is a subgroup of G if and only if for any $a, b \in H$, $ab^{-1} \in H$.*

Proof.

($\Rightarrow$) Assume that $H \leq G$ and $a, b \in H$. Since $b \in H$ and H is a group then $b^{-1} \in H$. H is a group so it must satisfy the closed property of the operation. Thus ab^{-1} must be in H.

($\Leftarrow$) Conversely, let $H \subseteq G$ such that $H \neq \phi$ and for all $a, b \in H$, there is always $ab^{-1} \in H$. We want to show that this will satisfy all 3 properties of a subgroup which are said before.

Since $H \neq \phi$, there exists some element $a \in H$. By the assumption $a, b \in H \Rightarrow ab^{-1} \in H$, choose $b = a$. So $ab^{-1} = aa^{-1} = e \in H$. Here a^{-1} must exist in G (since G is a group) and next we want to show that a^{-1} is in H also. Since $e, a \in H$ thus $ea^{-1} = a^{-1} \in H$. This implies that if $a, b, c, \ldots \in H$ then $a^{-1}, b^{-1}, c^{-1}, \ldots \in H$. For $a, b \in H$, we know that $b^{-1} \in H$ also. Thus $a\left(b^{-1}\right)^{-1} = ab \in H$. This concludes that H is a group.

Remark. Is it necessary to prove that $\circ$ satisfies associative property on H?

EXERCISE

1. Find all subgroups of S_3 and S_4.

2. Let H and K be subgroups of G. Determine whether $H \bigcup K$ is a subgroup of G or not? Give reason.

3. Let H and K be subgroups of G. Determine whether $H \bigcap K$ is a subgroup of G or not? Give reason.

4. Find all subgroups of $(\mathbb{Z}, +)$.

5. Let G be a group and $H = \{a \in G | (ax)^2 = (xa)^2, \forall x \in G\}$. Show that H is a subgroup of G.

Proof. First we want to prove that for any $a \in H$ then $a^{-1} \in H$ also. For any $x \in G$, $(x^{-1}a)^2(a^{-1}x)^2 = e$ and $(x^{-1}a)^2 = (ax^{-1})^2$ (because $a \in H$), it implies that

$$\begin{aligned}
(ax^{-1})^2(a^{-1}x)^2 &= e, \\
(xa^{-1})^2(ax^{-1})^2(a^{-1}x)^2 &= (xa^{-1})^2 e, \\
e(a^{-1}x)^2 &= (xa^{-1})^2, \\
(a^{-1}x)^2 &= (xa^{-1})^2, \qquad \forall x \in G.
\end{aligned}$$

This shows the result that we want. Let $a, b \in H$. We want to show that $ab^{-1} \in H$. For any $x \in G$,

$$\begin{aligned}
\left((ab^{-1})x\right)^2 &= \left(a(b^{-1}x)\right)^2 && \text{associative} \\
&= \left((b^{-1}x)a\right)^2 && a \in H \\
&= \left(b^{-1}(xa)\right)^2 && \text{associative} \\
&= \left((xa)b^{-1}\right)^2 && \text{if } b \in H \text{ then } b^{-1} \in H \\
&= \left(x(ab^{-1})\right)^2 && \text{associative.}
\end{aligned}$$

This shows that if $a, b \in H$ then ab^{-1} has the same property, i.e. $ab^{-1} \in H$. By theorem 1.4, it will be able to conclude that H is a subgroup. This proves the problem.

Theorem 1.5. *Let G be a group. Consider $\mathcal{S}$, which is a collection of some subgroups H. If $\mathcal{S} \neq \phi$ and*

$$\mathcal{H}^* = \bigcap_{H \in \mathcal{S}} H,$$

$\mathcal{H}^*$ is also a subgroup of G.

Proof. It is obvious that $\mathcal{H}^* \subseteq G$. Since an identity of G is in all subgroups of G, thus $\mathcal{H}^*$ is not an empty set. If $a, b \in \mathcal{H}^*$, it must be $a, b \in H, \forall H \leq G$. Since $H \leq G$, then $ab^{-1} \in H, \forall H \in G$. Thus $ab^{-1} \in \mathcal{H}^*$ also. By theorem 1.4, we can conclude that $\mathcal{H}^* \leq G$.

Theorem 1.6. *Let S be a* ***subset*** *of a group G and $\mathcal{S}$ is a collection of* ***all subgroups of*** *G* ***which contain*** *S. Define*

$$\langle S \rangle = \bigcap_{H \in \mathcal{S}} H. \tag{1.1}$$

Hence $\langle S \rangle$ must be a unique smallest subgroup of G which contains S.

Remark. $\langle S \rangle$ is a unique smallest subgroup of G which contains S means

1. $S \subseteq \langle S \rangle$.

2. $\langle S \rangle \leq G$.

3. If H is any subgroup of G which contains S then $\langle S \rangle \subseteq H$.

Proof. The proof is divided into 4 parts.

- $\langle S \rangle$ is a subgroup of G.

  Since $G \in \mathcal{S}$, thus $\mathcal{S}$ is not an empty set. By theorem 1.5, it shows that $\langle S \rangle \leq G$.

- $S \subseteq \langle S \rangle$.

  For any subgroup H which contains S, $H \in \mathcal{S}$. By the definition of $\langle S \rangle$, the intersection of all sets in $\mathcal{S}$ which contain S must contains S also. So $\langle S \rangle$ contains S.

- If H is any subgroup of G which contains S then $\langle S \rangle \subseteq H$.

  Since $\langle S \rangle$ is defined as in (1.1), for any $a \in \langle S \rangle$ then $a \in H, \forall H \in \mathcal{S}$. By the definition of subset, this means $\langle S \rangle \subseteq H, \forall H \in \mathcal{S}$.

- $\langle S \rangle$ is unique.

  Let $\langle S \rangle$ and $\langle S' \rangle$ be defined as in (1.1). If we consider $\langle S' \rangle$ as a set which contains S, thus $\langle S' \rangle \subseteq \langle S \rangle$. Conversely, if we consider $\langle S \rangle$ as a set which contains S, $\langle S \rangle \subseteq \langle S' \rangle$. That is $\langle S \rangle = \langle S' \rangle$.

Definition 1.7. Let $\langle S \rangle$ be defined as in theorem 1.6. We call $\langle S \rangle$ "**a subgroup of** G **generated by** S", or "S **generates the subgroup** $\langle S \rangle$".

In that case that S is finite, e.g. $\{a_1, a_2, \ldots, a_n\}$, we denote $\langle S \rangle$ by $\langle a_1, a_2, \ldots, a_n \rangle$.

If G is a group and $a \in G$, for any positive integer n, we give the definition of a^n and a^{-n} by the followings:

1. $a^0 = e$, (an identity of group G.) and $a^1 = a$.

2. $a^n = (a^{n-1})\, a$ and $a^{-n} = (a^{-1})^n$.

3. $a^m a^n = a^{m+n} = a^{n+m} = a^n a^m$.

4. $(a^m)^n = a^{mn} = (a^m)^n$.

Theorem 1.7. *If G is a group and $a \in G$, then*

$$\langle a \rangle = \{a^n\} \leq G,$$

where $n \in \mathbb{Z}$.

Proof. The proof of this theorem is left to be an exercise.

Example 1.14. In this example, we want to show some subgroups generated by some subsets : Consider $\langle 9, 12\rangle$, it is easy to see that

$$\begin{aligned}
\langle 9,12\rangle = \{\dots,-9,-6,-3,0,3,6,9,\dots\} &\leq (\mathbb{Z},+),\\
\langle 5,3\rangle = \{\dots,-3,-2,-1,0,1,2,3,\dots\} &= (\mathbb{Z},+),\\
\langle -1\rangle = (\mathbb{Z},+) &\leq (\mathbb{Q},+),\\
\langle -1\rangle = \{-1,1\} &\leq (\mathbb{Q}^*,\cdot),\\
\langle -1\rangle = \{-1,1\} &\leq (\mathbb{R}^*,\cdot),\\
\langle \imath\rangle = \{-\imath,\imath,-1,1\} &\leq (\mathbb{C}^*,\cdot),\\
\langle (123)\rangle = \langle (132)\rangle = \{(1),(123),(132)\} &\leq (S_3,\circ).
\end{aligned}$$

1.5 Cyclic Group and Elementary Properties

Definition 1.8 (Cyclic Subgroup $\langle a\rangle$)**.** The group H of theorem 1.7 is the **cyclic subgroup of** G **generated by** a, and will be denoted by $\langle a\rangle$.

Definition 1.9 (Generator; Cyclic group)**.** An element a of a group G **generates** G and is a **generator for** G if $\langle a\rangle = G$. A group G is **cyclic** if there is some element a in G that generates G.

Example 1.15. These are some well known examples.

- $(\mathbb{Z}_4,+)$ is cyclic and both 1 and 3 are generators, that is

$$< \bar{1} >=< \bar{3} >= \mathbb{Z}_4.$$

- $(\mathbb{Z},+)$ is cyclic and 1 is a generator, that is

$$< 1 >= \mathbb{Z}.$$

- $(\mathbb{R},+)$ is not cyclic.

- $(\mathbb{S}_3, +)$ is not cyclic.

Definition 1.10. An element a of a group G is of finite **order** m if m is the smallest positive integer such that $a^m = e$. Otherwise, we say that a is of **infinite order**.

Question : What is the order of e?

Theorem 1.8. *A subgroup of a cyclic group is cyclic.*

Proof. Let G be a cyclic group generated by a and let H be a subgroup of G. If $H = \{e\}$, then $H = \langle e \rangle$ is cyclic. If $H \neq \{e\}$, then $a^n \in H$ for some $n \in \mathbb{Z}^+$. Let m be the smallest integer in $\mathbb{Z}^+$ such that $a^m \in H$. We claim that $c = a^m$ generates H; that is,

$$H = \langle a^m \rangle = \langle c \rangle .$$

We must show that every $b \in H$ is a power of c. Since $b \in H$ and $H \leq G$, we have $b = a^n$ for some n. Find q and r such that

$$n = mq + r \qquad \text{for} \quad 0 \leq r < m$$

in accord with the division algorithm. Then

$$a^n = a^{mq+r} = (a^m)^q \, a^r,$$

so

$$a^r = (a^m)^{-q} \, a^n.$$

Now since $a^n, a^m \in H$, and H is a group, both $(a^m)^{-q}$ and a^n are in H. Thus

$$(a^m)^{-q} \, a^n; \qquad \text{that is, } a^r \in H.$$

Since m was the smallest positive integer such that $a^m \in H$ and $0 \leq r < m$, we mush have $r = 0$. Thus $n = qm$ and

$$b = a^n = (a^m)^q = c^q,$$

so b is a power of c.

Theorem 1.9. *Every cyclic group is abelean.*

Proof. Let G be a cyclic group, then $G = \langle a \rangle$. If $g_1, g_2 \in G$ thus there exist positive integers r and s such that

$$g_1 = a^r \qquad \text{and} \qquad g_2 = a^s.$$

Hence $g_1 g_2 = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = g_2 g_1$.

Remark. The converse of the theorem is not true. For example $(\mathbb{Q}, +)$ is an abelean group but it is not cyclic.

Theorem 1.10. *If G is an infinite cyclic group, G has two generators.*

Proof. Exercise.

Exercise

- Let G be a group and $a \in G$. Prove that an order of a is equal to an order of a^{-1}
- Prove that $(\mathbb{Q}, +)$ is not an cyclic group.
- Let $o(a)$ be an order of a. Suppose that $o(a) = n$ and GCD of m and n is 1. Show that $o(a^m) = n$.

1.6 Cosets

Definition 1.11 (Cosets)**.** Let H be a subgroup of G and $a \in G$. The subset

$$aH = \{ah | h \in H\}$$

is the **left coset of H in G containing** a, while

$$Ha = \{ha | h \in H\}$$

is the **right coset of H in G containing** a.

Example 1.16. Consider $3\mathbb{Z} = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\} \leq \mathbb{Z}$. These are left cosets of $3\mathbb{Z}$ in $\mathbb{Z}$

- $0 + 3\mathbb{Z} = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$,
- $1 + 3\mathbb{Z} = \{\dots, -8, -5, -2, 1, 4, 7, 10, \dots\}$,
- $2 + 3\mathbb{Z} = \{\dots, -7, -4, -1, 2, 5, 8, 11, \dots\}$.

. Also these are right cosets of $3\mathbb{Z}$ in $\mathbb{Z}$

- $3\mathbb{Z} + 0 = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$,
- $3\mathbb{Z} + 1 = \{\dots, -8, -5, -2, 1, 4, 7, 10, \dots\}$,
- $3\mathbb{Z} + 2 = \{\dots, -7, -4, -1, 2, 5, 8, 11, \dots\}$.

. We can see that both left and right cosets are identical, $i + 3\mathbb{Z} = 3\mathbb{Z} + i, i = 0, 1, 2$.

However for $H = \{(1), (12)\} \leq S_3$, some left and right cosets are

- $(13) \circ H = \{(13), (123)\}$,
- $H \circ (13) = \{(13), (132)\}$.

This example shows that $(13) \circ H \neq H \circ (13)$.

We denote G/H to be a collection of all left cosets of H in G

$$G/H = \{aH | a \in G\} \tag{1.2}$$

and denote $H \backslash G$ to be a collection of all right cosets of H in G

$$H \backslash G = \{Ha | a \in G\}. \tag{1.3}$$

Definition 1.12 (Normal Subgroup)**.** Let G be a group and $H \leq G$. If $aHa^{-1} = H$ for all $a \in G$, we call H **a normal subgroup of** G , this may be denoted by $H \triangleleft G$.

Theorem 1.11. *Let $H \leq G$. H is a normal subgroup of G if and only if $aHa^{-1} \subseteq H$ for all $a \in G$.*

Proof. Exercise.

Theorem 1.12. *Every subgroup of an abelean group is a **normal subgroup**.*

Proof. Exercise.

Theorem 1.13. *If $H \triangleleft G$ then $aH = Ha, \forall a \in G$. (On the other word, if H is a normal subgroup of G then $G/H = H \backslash G$.*

Proof. Exercise.

Theorem 1.14. *Let H be a normal subgroup of G and G/H be a set of left cosets of H in G, which is defined in (1.2). Definite the binary operator $\cdot$ on G/H by*

$$aH \cdot bH = abH.$$

$(G/H, \cdot)$ must be a group.

Proof.

- Claim that the binary operator $\cdot$ is well defined. Let $a, b, c, d \in G$ such that $aH = cH$ and $bH = dH$. Hence

  $$abH = a(bH) = a(dH) = a(Hd) = (aH)d = (cH)d = c(Hd) = c(dH) = cdH.$$

- (Associative)

  $$aH \cdot (bH \cdot cH) = aH(bcH) = a(bc)H = (ab)cH = abH \cdot cH = (aH \cdot bH) \cdot cH.$$

- (Identity) Since $H = eH$ thus

$$\begin{aligned} aH \cdot H &= (ae)H = aH \\ H \cdot aH &= (ea)H = aH. \end{aligned}$$

This is H is an identity of G/H.

- (Inverse) Let aH be any element of G/H. So

$$\begin{aligned} aH \cdot a^{-1}H &= (aa^{-1})H = eH = H \\ a^{-1}H \cdot aH &= (a^{-1}a)H = eH = H. \end{aligned}$$

We can say that $(aH)^{-1} = a^{-1}H$. All of these give a proof of the theorem.

Definition 1.13 (Factor Group)**.** We call $(G/H, \cdot)$ which is defined in theorem 1.14 that **a quotient group of** G **by** H or **a factor group of** G **by** H.

Example 1.17. $5\mathbb{Z} \lhd \mathbb{Z}$ and

$$\mathbb{Z}/\mathbb{Z}_5 = \{0 + 5\mathbb{Z}, 1 + 5\mathbb{Z}, 2 + 5\mathbb{Z}, 3 + 5\mathbb{Z}, 4 + 5\mathbb{Z}\}.$$

This is obvious that $(\mathbb{Z}/\mathbb{Z}_5, +)$ is a group.

Exercise

1. Let G be a group and $H \leq G$ which is $a, b \in G$. Show that
   (a) $a^{-1}b \in H \iff aH = bH$.
   (b) $a \in H \iff aH = H$.
2. Let G be a group and H, K are subgroups of G. Show that $a\,(H \cap K) = aH \cap aK, \forall a \in G$.
3. Let $H \lhd G$ and $K \lhd G$. Show that
   (a) $H \cap K \lhd G$.
   (b) $HK \lhd G$ where $HK = \{ab | a \in H \text{ and } b \in K\}$

1.7 Homomorphisms and Isomorphisms

Definition 1.14 (Homomorphism)**.** A map φ of a group G into a group G' is a **homomorphism** if

$$\varphi(ab) = \varphi(a)\varphi(b)$$

for all $a, b \in G$.

Example 1.18.

1. Let φ be a mapping from a group G to a group G' which is defined by

$$\varphi(a) = e_{G'}, \qquad \forall a \in G,$$

where $e_{G'}$ is an identity in a group G'.

This is easy to see that φ is a homomorphism.

2. Let G be an **abelean** group and $\varphi : G \to G$ defined by

$$\varphi(a) = a^n, \qquad \forall a \in G,$$

where n is some integer. It is found that

$$\varphi(ab) = (ab)^n = a^n b^n = \varphi(a)\varphi(b), \qquad \forall a, b \in G.$$

This shows that φ is a homomorphism.

Theorem 1.15. *Let G and G' be groups. If $\varphi : G \to G'$ is a homomorphism. The followings must satisfy :*

1. *$\varphi(e_G) = e_{G'}$, where e_G is an identity of G and $e_{G'}$ is an identity of G'.*

2. *$\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in G$.*

3. *If $K \leq G$ then $\varphi(K) \leq G'$, where $\varphi(K) = \{\varphi(a) | a \in K\}$. We call $\varphi(K)$ **a homomorphic image of** K.*

Proof. Exercise.

Definition 1.15 (Kernel)**.** If G and G' are groups and $\varphi : G \to G'$ is homomorphism. We define

$$\ker(\varphi) = \{x \in G | \varphi(x) = e_{G'}\}.$$

we call $\ker(\varphi)$ **a kernel of** G.

Remark. $\ker(\varphi) \neq \phi$. By theorem 1.15, we know that $e_G \in \ker(\varphi)$

Definition 1.16 (Isomorphism)**.** A homomorphism $\varphi : G \to G'$ is call **an isomorphism** if φ is an bijective function from G to G'. If there exists an isomorphism mapping G to G', we say that G **is isomorphic to G'**. We may sometimes denote it by $G \approx G'$.

Example 1.19. These are some examples of isomorphic groups.

- $(\mathbb{R}, +) \approx (\mathbb{R}^+, \cdot)$ (Exercise)
- $(\mathbb{Z}_4, +) \approx (\mathbb{U}_5, \cdot)$ (Exercise)

Theorem 1.16. *For two groups which are isomorphic, if one group satisfies the following properties, the other group must satisfy also :*

1. *abelean.*
2. *cyclic.*
3. *have subgroup which is order n.*
4. *have an element which is order n.*
5. *each element is an inverse of itself.*
6. *each element has finite order.*

etc.

Proof. Exercise.

Theorem 1.17. *Let G be a cyclic group.*

1. *If G is an infinite group then $G \approx \mathbb{Z}$.*
2. *If G is a finite group with order n then $G \approx \mathbb{Z}_n$.*

Proof. Exercise.

Theorem 1.18. *Let G be a finite group with order n. G must isomorphic to some subgroup of S_n.*

Proof. Exercise.

Exercise

1. Prove that
   (a) If G and H are groups, thus $G \times H \approx H \times G$.
   (b) If $G_1 \approx G_2$ and $H_1 \approx H_2$ are groups, then $G_1 \times H_1 \approx G_2 \times H_2$.
2. Prove that
   (a) $G \approx G$.
   (b) $G \approx G'$ then $G' \approx G$.
   (c) $G \approx G'$ and $G' \approx G''$ implies $G \approx G''$.

1.8 Rings

Definition 1.17 (Rings)**.** We call $(R, +, \cdot)$ **ring**, if R is not empty, both $+$ and $\cdot$ are binary operation on R, $+$ is called **additive operation** and $\cdot$ is called **multiplicative operation**, and R satisfies the following properties :

1. $(R,+)$ is **an abelean group**.

2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for all $a, b, c \in R$ (Multiplication is associative.)

3. For all $a, b, c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad \text{(left distributive rule.)}$$

$$(a + b) \cdot c = a \cdot c + b \cdot c, \qquad \text{(right distributive rule.)}$$

Example 1.20. These are some well known rings :

- $(\mathbb{Z}, +, \cdot), (\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are all rings.

- $(k\mathbb{Z}, +, \cdot)$ where k is an integer number, is a ring.

- $(\mathbb{Z}_n, +, \cdot)$ where $n = 2, 3, \dots$, is a ring.

- $(\mathcal{M}_{n\times n}, +, \times)$ is also a ring.

- Let F be a set of all function $f : \mathbb{R} \to \mathbb{R}$

Theorem 1.19. *If R is a ring and 0 is an zero identity of R with respect to $+$ and $a, b, c \in R$ then the following statements are true.*

1. $0a = a0 = 0$

2. $a(-b) = (-a)b = -(ab)$

3. $(-a)(-b) = ab$

4. $a(b - c) = ab - ac$ *and* $(a - b)c = ac - bc$

Proof. Exercise.

Definition 1.18 (A ring with unity)**.** Let $(R, +, \cdot)$ be a ring. We call R **a ring with unity** if R has an identity with respect to the multiplication operation $\cdot$.

Definition 1.19 (Commutative ring)**.** Let $(R, +, \cdot)$ be a ring. We call R **a commutative ring** if $ab = ba$ for all $a, b \in R$. If $ab \neq ba$ for some $a, b \in R$, we call R **a noncommutative ring**.

Definition 1.20 (Unit)**.** Let $(R, +, \cdot)$ be **a ring with unity** and $a \in R$. If a has an inverse with respect to the multiplicative operation $\cdot$, we call a **a unit of** R and call its inverse (with respect to the multiplicative operation $\cdot$) **a reciprocal of** a.

Theorem 1.20. *Let $(R, +, \cdot)$ be a ring with unity which is not a trivial ring. We find that **the identity** 0 **is not equal to the unity** 1.*

Proof. Exercise.

Exercise

1. Let R and S be rings and $R \times S$ be an cartesian product of R and S. For (r, s) and $(r', s') \in R \times S$, define

$$\begin{aligned} (r,s) + (r',s') &= (r + r', s + s'), \\ (r,s) \cdot (r',s') &= (r \cdot r', s \cdot s'). \end{aligned}$$

   Show that $(R \times S, +, \cdot)$ is a ring.

2. Define operations $*$ and $\circ$ on $\mathbb{Q}$ by

$$\begin{aligned} a * b &= a + b + 1, \\ a \circ b &= ab + a + b, \end{aligned}$$

   for $a, b \in \mathbb{Q}$. Show that $(\mathbb{Q}, *, \circ)$ is a ring.

3. Prove theorem 1.19.

4. Prove theorem 1.20.

1.9 Fields and Ordered Fields

Definition 1.21 (Field)**.** Let $(F, +, \cdot)$ be a nontrivial commutative ring with unity. If $F\backslash\{0\}$ is a group with respect to the multiplicative operation $\cdot$ then we call F **a field**.

Remark. To make the definition of a field looked like definitions of a group and a ring as before, we make define a field by the following statements :

We call $(F, +, \cdot)$ **a field** if F is not an empty set and $+$ and $\cdot$ are binary operations on F which satisfy the following properties :

1. $(F, +)$ is an abelean group.
2. $(F\backslash\{0\}, \cdot)$ is an abelean group.
3. For all $a, b, c \in F$

$$a \cdot (b + c) = a \cdot b + a \cdot c \qquad \text{and} \qquad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Example 1.21. $\mathbb{Z}$ is not a field because $2 \in \mathbb{Z}$ but 2 has no an inverse with respect to an operation multiplication. However, it is obvious that $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields.

$(\mathcal{M}_{n\times n}, +, \cdot)$, where $\mathcal{M}_{n\times n}$ is a set of all $n \times n$ nonsingular matrices, is not a field even $\mathcal{M}_{n\times n}$ has a unity and for all $A \in \mathcal{M}_{n\times n}$ such that A is not a zero matrix has an inverse. Why?

A **field** provides an algebraic structure which makes us enable to have basic operations of addition and multiplication. Furthermore, there is superimposed on this algebraic structure an order relation.

Definition 1.22 (Ordered Field)**.** Let $(F, +, \cdot)$ be a field. A field F is **an ordered field** provided that there is a nonempty subset P of F such that the following two postulates hold:

1. For each $a \in F$, one and only one of the following conditions holds:

   - $a \in P$, or
   - $a = 0$, or
   - $-a \in P$.

2. If $a, b \in P$, then $a + b \in P$ and $ab \in P$.

In an ordered field F, the elements of the set P are called **positive**. All other elements of F, except zero, are called **negative**. If two elemtents of F are both positive, or both negative, then we say they have the **same sign**; if one is positive and the other is negative, they have **opposite signs**.

Set of rational numbers $(\mathbb{Q}, +, \cdot)$ provides an ordered field and set of real numbers $(\mathbb{R}, +, \cdot)$ does also; however set of complex numbers $(\mathbb{C}, +, \cdot)$ does not. Inequalities can be introduced in any ordered field F. For $a, b \in F$, we make the following definitions:

Definition 1.23 (Inequalities).

1. $a < b$ (read, "a is less than b") provided $b - a$ is positive.
2. $a > b$ (read, "a is greater than b") provided $b < a$ or $a - b$ is positive.
3. $a \leq b$ (read, "a is less than or equal to b") provided $a < b$ or $a = b$.
4. $a \geq b$ (read, "a is greater than or equal to b") provided $a > b$ or $a = b$.

Exercise

1. Let F be an ordered field and $a \in F$. Show that

   (a) a is positive if and only if $a > 0$.

(b) a is negative if and only if $-a$ is positive.

2. Let F be an ordered field and $a, b, c \in F$. Show that

   (a) $a < b$ if and only if $a + c < b + c$.

   (b) If $a \leq b$ and $c \leq d$ then $a + c < b + d$.

   (c) If $a < b$ and c is positive then $ac < bc$.

   (d) If $a < b$ and c is negative then $ac > bc$.

3. (Laws of sign.) Let a and b be elements of an ordered field F. Show that if a and b have the same sign, then their product is positive; if they have opposite signs, then their product is negative.

4. Show that the sum of a positive real number and its reciprocal is greater than or equal to 2.

Chapter 2

Real numbers

2.1 Real numbers

Remark. Why is the field of rational numbers $\mathbb{Q}$ not sufficient?

2.1.1 Geometric Motivation

For a long time, the Greeks thought that every number is **rational**. A rational number $\dfrac{m}{n}$ can be represented geometrically as follows :

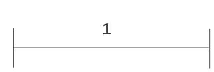


Figure 2.1: Line segment with length 1.

Fix a line segment, give it length 1. Divide it into n pieces of equal length. Each piece will have length $\dfrac{1}{n}$ Past m such segments together. The resulting segment has length $\dfrac{m}{n}$.

With the discovery of the law of Pythagoras, a problem arose : By the law of Pythagoras,

$$x^2 = 1^2 + 1^2 = 2,$$

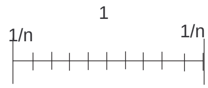


Figure 2.2: Line segment with length 1 divided into n pieces of equal length.

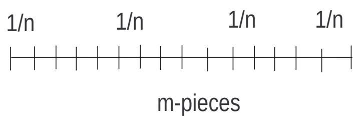


Figure 2.3: m $\frac{1}{n}$-pieces past together.

where x is the length of the *hypothenuse*.

Question : What rational number is x?

Theorem 2.1. *There exists* ***no*** *rational number x such that $x^2 = 2$.*

Proof. Assume to the contrary that there exists $x \in \mathbb{Q}$, $x = \dfrac{m}{n}$ so that $x^2 = 2$. we assume that GCD of m and n is 1.

$$\begin{aligned}
\frac{m^2}{n^2} &= 2 \\
m^2 &= 2n^2 \\
\text{If } 2|m^2 \text{ then } 2|m \text{ and implies to } m &= 2k \text{ for some integer } k. \\
(2k)^2 &= 2n^2 \\
4k^2 &= 2n^2 \\
2k^2 &= n^2
\end{aligned}$$

This means $2|n^2$ and $2|n$. Thus the GCD of n and m cannot be 1. It contradicts to the assumption. x cannot be rational.

Because $x^2 = 2$ has no solution in $\mathbb{Q}$, we need to introduce additional numbers. Before we must discuss the idea of upperbounds.

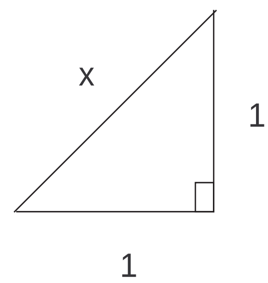


Figure 2.4: Figure of a righted-triangle which has the length of sides 1,1 and x.

2.1.2 Upper and Lower Bound

Definition 2.1 (Upper and Lower Bound)**.** Let F be an ordered set with order " $<$ ", and S be a nonempty subset of F.

- an element M of F is called **upper bound of** S if

$$x \leq M, \qquad \text{for all } x \in S.$$

- an element m of F is called **lower bound of** S if

$$m \leq x, \qquad \text{for all } x \in S.$$

- If an upper bound M of S exists, then we say that S is **bounded above**.
- If a lower bound m of S exists, then we say that S is **bounded below**.
- If both, upper and lower bounds exists, then we say that S is **bounded**.
- If S is not bounded, then we say that S is **unbounded**.

Example 2.1. 1. Let $F = \mathbb{Q}$ and $S = \{x \in \mathbb{Q} : 0 < x < 1\}$.

- **Upper bound** : 1,2,2.5,... Every rational number $M \geq 1$ is an upper bound.

- **Lower bound** : 0,-1,-4.25,... Every rational number $m \leq 0$ is an lower bound.

This shows that S is bounded.

2. Let $F = \mathbb{Q}$ and $S = \{x \in \mathbb{Q} : x > 0\}$.

   - **Lower bound** : Every rational number $m \leq 0$ is an lower bound.
   - **Upper bound** : There is no upper bound. This shows that S is unbounded.

3. Let $F = \mathbb{Q}$ and $S = \mathbb{Q}$. $S = \mathbb{Q}$ is unbounded. (Exercise)

Let F be a field. Given a subset S of F, we set

$$-S = \{y \in F : y = -x, \text{ for some } x \in S\}$$

Theorem 2.2. *Let S be a nonempty subset of an ordered field F. Let $m, M \in F$*

1. *M is an upper bound of S iff $-M$ is a lower bound of $-S$.*
2. *m is a lower bound of S iff $-m$ is an upper bound of $-S$.*

Proof.

1. ($\Rightarrow$) Suppose, M is an upper bound of S

$$\begin{aligned} M &\geq x, \qquad \forall x \in S \\ -M &\leq -x, \qquad \forall x \in S \\ -M &\leq y, \qquad \forall y \in -S \end{aligned}$$

   thus $-M$ is a lower bound of $-S$.

   ($\Leftarrow$) Just go backward above.

2. Similar.

Let S be a nonempty subset of an ordered field F. An element $l \in F$ is called **least upper bound** or **supremum** of S if

1. l is an upper bound of S.

2. If M is **any** upper bound of S, then $l \leq M$.

We write $l = \sup S$ ($l =$ l.u.b S)

If in addition, l is an element of S, then l is called the **maximum** of S.

Remark. There exists at most one least upper bound. We speak of **the** least upper bound.

An element $g \in F$ is called **greatest lower bound** or **infimum** of S if

1. g is a lower bound of S.

2. If m is **any** lower bound of S, then $m \leq g$.

We write $g = \inf S$ ($g =$ g.l.b S)

If in addition, g is an element of S, then g is called the **minimum** of S.

Example 2.2. $S = \{x \in \mathbb{Q}, 0 < x < 1\}$.

Claim : $1 = \sup S$

1. For all x in S, $x < 1 \Rightarrow 1$ is an upper bound.

2. Let M be any upper bound. Must show that $1 \leq M$.

Suppose to the contrary that

$$\begin{array}{rcl} 0 < & M & < 1 \\ 0 < & \frac{M}{2} & < \frac{1}{2} \\ 0 < \frac{1}{2} < & \frac{M}{2} + \frac{1}{2} & < 1 \end{array}$$

$$\Rightarrow \quad \frac{M+1}{2} \in S.$$

$$\begin{array}{rcl} \text{but} \quad \frac{M}{2} + \frac{M}{2} & < & \frac{M}{2} + \frac{1}{2} \\ M & < & \frac{M+1}{2} \in S \end{array}$$

This shows M is less than some element in S. It contradicts to the assumption that M is an upper bound of S. $M < 1$ is impossible. Thus $M \geq 1$.

$\Rightarrow$ 1 is least upper bound.

Similarly $0 = \inf S$.

Example 2.3. $S = \{x \in \mathbb{Q}, 0 \leq x \leq 1\}$. Just as before $\sup S = 1$ and $\inf S = 0$. Moreover, $1 \in S \Rightarrow 1$ is the maximum of S and $0 \in S \Rightarrow 0$ is the minimum of S

Exercise

1. $S = \{\frac{1}{n} | n = 1, 2, 3, \ldots\}$. Show that $\inf S = 0$ and S has no minimum.

2. Let S be a nonempty subset of an ordered field F. Show that

   (a) If $\sup S$ exists, say $l = \sup S$ then $\inf(-S)$ exists and $\inf(-S) = -l$.

   (b) If $\inf S$ exists, say $g = \inf S$ then $\sup(-S)$ exists and $\sup(-S) = -g$.

3. Let $S \subseteq \mathbb{Q}$ and $S = \{x | x^2 < 2\}$. Does the supremum of S exist? Give a reason.

Theorem 2.3. *Let S be a nonempty subset of an ordered field F. And element $l \in F$ is the least upper bound of S iff*

1. *l is an upper bound.*

2. *For any $\varepsilon > 0$, there exists $x \in S$ so that $l - \varepsilon < x$.*

Proof. ($\Rightarrow$) Assume $l = \sup S$. Clearly, l is an upper bound of S. Let $\varepsilon > 0$ be given then $l - \varepsilon$ cannot be an upper bound. This means, $\exists x \in S$ so that $l - \varepsilon < x$.

($\Leftarrow$) Suppose $l \in F$ satisfies (1) and (2). Suppose to the contrary that there exists an upper bound M of S with $M < l$. Let $\varepsilon = l - M > 0$. By (2), $\exists x \in S$ so that $l - \varepsilon < x$. $l - (l - M) < x \Rightarrow M < x$. It contradicts to the assumption. So every upper bound M of S satisfies $l \leq M$. Hence l is the least upper bound of S.

Given two subsets S, T of a field F, we set

$$S + T = \{s + t | s \in S, t \in T\}.$$

If $c \in F$, we set

$$cS = \{cs | s \in S\}.$$

Theorem 2.4. *If $s = \sup(S)$ and $t = \sup(T)$ then $S + T$ has the least upper bound $\sup(S + T) = s + t$.*

Proof. Exercise. (Hint. apply theorem 2.3 to prove this theorem.)

Theorem 2.5. *Let S be a nonempty subset of an ordered field F, and $c \in F$. Suppose $s = \sup(S)$ exists.*

1. *If $c > 0$ then $\sup(cS)$ exists and $\sup(cS) = cs$.*

2. *If $c < 0$ then $\inf(cS)$ exists and $\inf(cS) = cs$.*

Proof.

1. Assume $c > 0$

(a) Since $s = \sup(S)$

$$\begin{aligned} x &\leq s && \forall x \in S \\ cx &\leq cs && \text{as } c > 0 \\ y &\leq cs && \forall y \in cS \end{aligned}$$

So cs is an upper bound of cS.

(b) Because $s = \sup(S)$, by theorem 2.3, for arbitrary $\varepsilon > 0$, we can find $x \in S$ so that

$$\begin{aligned} s - \frac{\varepsilon}{c} &< x \\ cs - \varepsilon &< cx && \forall x \in S \\ cs - \varepsilon &< y && \forall y \in cS \end{aligned}$$

By theorem 2.3 again, $cs = \sup(cS)$

2. $c < 0$

(a) assume $c = -1$. So $cS = -1S = -S$. By theorem 2.2, $\inf(cS) = \inf(-S) = -\sup(S) = -s = cs$

(b) In general, we write $c = -\tilde{c}$ where $\tilde{c} > 0$.

$$\begin{aligned} cS &= (-\tilde{c})S = -(\tilde{c}S) \\ \inf(cS) &= \inf(-(\tilde{c}S)) \\ \text{By case 2a.} \qquad \inf(-(\tilde{c}S)) &= -\sup(\tilde{c}S) \\ \text{Since } \tilde{c} > 0 \qquad \sup(\tilde{c}S) &= \tilde{c}s \\ \text{Thus} \qquad \inf(-(\tilde{c}S)) &= -\tilde{c}s = cs \\ \therefore \qquad \inf(cS) &= cs \end{aligned}$$

Remark. Theorem 2.3,2.4,2.5 can be reformulated for greatest lower bounds (infimums) in a very similar way.

Question : Does every subset S of $\mathbb{Q}$ which is bounded above have the least upper bound?

Theorem 2.6. *Let $S = \{x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2\}$. The set S is bounded above but $\sup(S)$ does not exist.*

Proof.

1. Since there are many rational numbers a such that $a^2 > 2$, e.g 1.5,2,3,.... It is obvious that S is bounded above.

2. Now show that S has **no** least upper bound.

   Claim : If l is the least upper bound of S then $l^2 = 2$.

   Proof of the claim : Suppose $l = \sup(S)$ exists. Consider the number

$$q = l - \frac{l^2 - 2}{l + 2} = \frac{2l + 2}{l + 2} \tag{2.1}$$

$$q^2 - 2 = \left(\frac{2l + 2}{l + 2}\right)^2 - 2 = \frac{2(l^2 - 2)}{(l + 2)^2} \tag{2.2}$$

   (a) Case 1. $l^2 < 2$

   As $l^2 - 2 < 0$ then $q^2 - 2 < 0$ and $q^2 < 2$ (by equation (2.2)). Since $q^2 < 2$ then $q \in S$. Equation 2.1 shows that $q > l$ which contradicts to the assumption that l is the least upper bound of S. So this case is impossible.

   (b) Case 2. $l^2 > 2$

   As $l^2 - 2 > 0$ then $q^2 - 2 > 0$ and $q^2 > 2$ (by equation (2.2)). Thus q is an upper bound of S. By Equation 2.1, $q < l$ this contradicts to l is the least upper bound which must $l \leq q$. So $l^2 > 2$ is also impossible.

Thus, necessarily, $l^2 = 2$. This proves the claim.

However, by theorem 2.1, the equation $l^2 = 2$ has no solution in $\mathbb{Q}$. This shows that S has **no** least upper bound in $\mathbb{Q}$ by the claim.

Remark. The proof shows the fact that S *has no least upper bound in* $\mathbb{Q}$ and the fact that $l^2 = 2$ *has no rational solution* are equivalent.

Definition 2.2 (Completeness Property)**.** An ordered field F has the **the least-upper-bound property** if **every** nonempty subset S of F which is bounded above has **a least upper bound**. The least-upper-bound property is also called **the completeness property** or **completeness axiom**.

Remark. $\mathbb{Q}$ does not have the completeness property as shown in theorem 2.6.

Theorem 2.7. *There exists an ordered field* $\mathbb{R}$ *which has the completeness property. Moreover, this field contains* $\mathbb{Q}$ *as a subfield.*

Proof. Coming soon in the seminar.

Remark. The elements of $\mathbb{R}$ are called **real numbers**. An element of $\mathbb{R}$ which is not rational is called **irrational**.

Theorem 2.8. *There exists* $x > 0$ *in* $\mathbb{R}$ *such that* $x^2 = 2$.

Proof. Set $S = \{x \in \mathbb{R} : x > 0, x^2 < 2\}$. As shown in the proof of theorem 2.6, S is bounded above. As $\mathbb{R}$ has the least-upper-bound property

$$l = \sup(S), \qquad l \in \mathbb{R} \qquad \text{So clearly } l > 0.$$

By the proof of the claim in part 2 of theorem 2.6, $l^2 = 2$, l is the solution of $x^2 = 2$. l may be sometimes denote by $\sqrt{2}$.

Furthermore, for every $x > 0$ in $\mathbb{R}$, and every $n \in \mathbb{Z}^+$ (or natural numbers $\mathbb{N}$), there exists a unique $y > 0$ so that $y^n = x$. This y is written as $y = \sqrt[n]{x}$ or $y = x^{1/n}$.

Exercise

1. Prove the theorem 2.4, page 39.

2. Prove or disprove each of the following. (The sets cS is as defined in Theorems 2.5)

   (a) If S is bounded, then every subset of S is bounded.

   (b) If every subset of S is bounded then S is bounded.

   (c) If c is a real number and S is bounded above, then cS is bounded above.

   (d) If c is a real number and S is bounded, then cS is bounded.

   (e) If c is a real number and cS is bounded, then S is bounded.

Theorem 2.9 (Archimedian Property)**.** *If* $a, b \in \mathbb{R}$, $a > 0$ *then there exists* $n \in \mathbb{Z}^+$ *(or* $\mathbb{N}$*) such that* $b < na$.

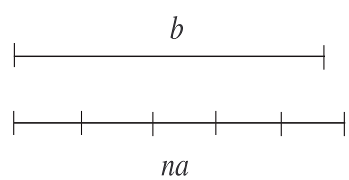


Figure 2.5: Figure to show an idea of Archimedian property.

Proof. Suppose to the contrary that $b \geq na$ for all $n \in \mathbb{N}$. Thus the set $S = \{na | n \in \mathbb{N}\}$ is bounded above (by b). By **the completeness property** $l = \sup(S)$ must exist. In particular $na \leq l$, for all $n \in \mathbb{N}$. Since $n \in \mathbb{N}$, this implies $n+1 \in \mathbb{N}$ also.

$$\begin{aligned} (n+1)a &\leq l \\ na + a &\leq l \\ na &\leq l - a \qquad \forall n \in \mathbb{N} \end{aligned}$$

So $l - a$ is an upper bound of S. $l - a < l$ contradicts to l is the least upper bound of S. Thus we must have $b < na$ for some $n \in \mathbb{N}$. This theorem is called **Archimedian property** or **Archimedian law**.

Corollary 2.10. *Set $\mathbb{Z}$ and $\mathbb{N}$ is **not bounded above** in $\mathbb{R}$.*

Proof. Suppose to the contrary that there exits $b \in \mathbb{R}$ such that $n \leq b$, for all $n \in \mathbb{Z}$ (or $\mathbb{N}$). By the Archimedian property ($a = 1$), there exists n_0 s.t. $b < n_0$ for some $n_0 \in \mathbb{Z}$ (or $\mathbb{N}$). This is contradiction. Thus $\mathbb{Z}$ (or $\mathbb{N}$) is unbounded in $\mathbb{R}$.

Corollary 2.11. *Let $\varepsilon > 0$ be any positive real number. Then there exists $n \in \mathbb{N}$ such that*

$$0 < \frac{1}{n} < \varepsilon$$

Proof. Exercise.

Theorem 2.12. *Let $x \in \mathbb{R}$ and $S = \{n \in \mathbb{Z} | n \leq x\}$ then there exists $m \in S$ so that $m \leq x < m + 1$.*

Proof. It is clear that S is bounded above (x is one of upper bounds). Thus $n_0 = \sup(S)$ exists in $\mathbb{R}$.

Claim : n_0 is an integer.

By theorem 2.3, there exists $m \in S$ such that $n_0 - \frac{1}{2} < m \leq n_0$. Thus $n_0 < m + \frac{1}{2} \leq m + 1$. So

$$m \leq n_0 < m + 1.$$

This inequality implies that m must be the largest element of S which means $m = n_0$, i.e. the maximum is equal to the supremum.

Remark. The integer m in the theorem such that $m \leq x < m + 1$ is called **the greatest integer less than or equal to** x. The new function may be defined as

$$\lfloor x \rfloor = m,$$

which is called **a floor function**. On the other way, if we define the new function

$$\lceil x \rceil = n,$$

where n is the integer such that $n - 1 < x \leq n$. This function is called **a ceil function**.

Exercise

1. Prove corollary 2.11

2. In the case that we consider on the set of rational numbers. Prove that if $a, b \in \mathbb{Q}, a > 0$ then there exists $n \in \mathbb{Z}^+$ such that $b < na$. This theorem looks like the **Archimedian property** which is true on the set of rational numbers also. However the proof of this theorem is not exactly the same with the prove of Archimedian property for a real number. Why?

Theorem 2.13 (Density of the rational numbers in $\mathbb{R}$)**.** *Let x, y be two real numbers with $x < y$ then there exists **a rational number** r such that $x < r < y$.*

Proof. Let $a = 1$ and $b = \frac{1}{y-x}$. By the Archimedian property, there exists $n \in \mathbb{Z}^+$ such that

$$\begin{aligned} \frac{1}{y-x} &< n \\ 1 &< ny - nx \\ nx + 1 &< ny \end{aligned}$$

By theorem 2.12, there exits $m \in \mathbb{N}$ such that $m \leq nx < m + 1$. Thus

$$m \leq nx < m + 1 \leq nx + 1 < ny.$$

Divide by n

$$x < \frac{m+1}{n} < y, \qquad \text{which } \frac{m+1}{n} \in \mathbb{Q}.$$

Theorem 2.14 (Density of the irrational numbers in $\mathbb{R}$)**.** *Let x, y be two real numbers with $x < y$ then there exists* ***an irrational number*** *q such that $x < q < y$.*

Proof. By theorem 2.13, there exists $r \in \mathbb{Q}$ so that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Thus $x < \sqrt{2}r < y$. If $r = 0$ then we have $0 < y$ and there must be another rational number r', so that

$$\frac{x}{\sqrt{2}} < 0 < r' < \frac{y}{\sqrt{2}}.$$

This shows that there must exist a nonzero rational number r such that

$$x < \sqrt{2}r < y.$$

Claim : $\sqrt{2}r$ is irrational if r is a nonzero rational number. Suppose to the contrary that $\sqrt{2}r$ is rational then $\sqrt{2}r = \frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. Then $\sqrt{2} = \frac{a}{br}$. Since r is rational then $r = \frac{\bar{a}}{\bar{b}}$ for some $\bar{a} \in \mathbb{Z}$ and $\bar{b} \in \mathbb{Z}^+$.

$$\sqrt{2} = \frac{a\bar{b}}{b\bar{a}}.$$

This is contradict to $\sqrt{2}$ is an irrational number. Hence $\sqrt{2}r$ where r is a nonzero rational number is irrational.

Let $q = \sqrt{2}r$ then we can find an irrational number q, which is $x < q < r$.

Exercise

1. Prove or disprove

   (a) If x and y are rational numbers, then $x + y$ is rational.

   (b) If x and y are rational numbers, then xy is rational.

(c) If x and y are irrational numbers, then $x + y$ is irrational.

(d) If x and y are irrational numbers, then xy is irrational.

(e) If x is rational but y is irrational, then $x + y$ is irrational.

(f) If x is rational ($x \neq 0$) but y is irrational, then xy is irrational.

2.2 Cardinality

Definition 2.3. A set $\mathcal{A}$ is said to be **cardinally equivalent** (or equinumerous) to a set B, if there exists a bijection

$$f : \mathcal{A} \xrightarrow[\text{onto}]{1-1} B.$$

We write $\mathcal{A} \sim B$

Example 2.4. • In the case that $\mathcal{A} = \{a, b, c\}$ and $B = \{1, 2, 3\}$. The map $f : \mathcal{A} \longrightarrow B$ defined by

$$\begin{aligned} f(a) &= 1, \\ f(b) &= 2, \\ f(c) &= 3. \end{aligned}$$

It is clear that f is a bijection. So $\mathcal{A} \sim B$.

• $U = \{x \in \mathbb{R} : 0 < x < 1\}$ and $I = \{x \in \mathbb{R} : 0 < x < 2\}$

Define $f : U \longrightarrow I$ by $f(x) = 2x$.

Claim : f is a bijection.

Injective Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in U$.

$$\begin{aligned} 2x_1 &= 2x_2 \\ x_1 &= x_2 (\Rightarrow) \text{ injective} \end{aligned}$$

Surjective Let $y \in I$ be arbitrary. Set $x = \frac{y}{2}$. Since $0 < y < 2 \Leftarrow 0 < x = \frac{y}{2} < 1$.

Thus $x \in U$. Then $f(x) = 2x = 2\frac{y}{2} = y$. $\Leftarrow$ f is surjective.

This proves the claim. Thus $U \sim I$.

Theorem 2.15. *Let A, B, C be sets.*

1. *If $A \sim B$ then $B \sim A$.* *(Symmetry)*

2. *$A \sim A$* *(Reflexive)*

3. *If $A \sim B$ and $B \sim C$ then $A \sim C$* *(Transitivity)*

Proof. Exercise.

Remark. A relation satisfies symmetry, reflexive and transitivity properties is called **an equivalent relation.**

Definition 2.4 (Finite and Infinite sets)**.** Given an $n \in \mathbb{Z}^+$, set $C_n = \{1, 2, 3, \ldots, n\}$. A set $\mathcal{A}$ is **finite** if it is **empty** or $\mathcal{A} \sim C_n$ for some $n \in \mathbb{Z}^+$. A set which is not finite is called **infinite**.

We say $\mathcal{A}$ has n elements if $\mathcal{A} \sim C_n$. We write $|\mathcal{A}| = n$. ($|\mathcal{A}|$ is called the cardinality of $\mathcal{A}$).

Theorem 2.16. *Let A, B be nonempty finite sets. $A \sim B$ iff $|A| = |B|$.*

Proof. $\Rightarrow$ Suppose $A \sim B$.

Let $n = |A|$. There exists a bijection $f : C_n \longrightarrow A$.

Because $A \sim B$, there exists a bijection $g : A \longrightarrow B$. Then $g \circ f$ is a bijection of C_n onto B. So $|B| = n$ also.

$\Leftarrow$ Suppose $|A| = |B|$. Let $n = |A| = |B|$. There exists bijections

$$f : C_n \Longrightarrow A$$

$$g : C_n \Longrightarrow B.$$

Then $g \circ f^{-1}$ is a bijection of A onto B. So $A \sim B$.

Example 2.5. We have already seen in some previous examples that $\mathbb{Z}_n, \mathbb{U}_n, S_n$ are finite sets while $\mathbb{Z}, k\mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are infinite sets. Furthermore,

$$\begin{aligned} |\mathbb{Z}_n| &= n, \\ |\mathbb{U}_n| &= \phi(n), \\ |\mathbb{S}_n| &= n!, \end{aligned}$$

where $\phi(n)$ is a number of positive integers a which GCD of a and n equals to 1.

However, even the cardinalities of some sets are infinite, it is important to have an to idea to make comparison of those sets. This concept will be shown later.

Exercise

1. Show that

   (a) $\mathbb{Z}^+ \sim \mathbb{Z}$.

   (b) $\mathbb{Z} \sim 2\mathbb{Z}$.

   (c) $(-1, 1) \sim (0, 1)$.

   (d) $(a, b) \sim (c, d)$.

   (e) $[0, 1] \sim (0, 1)$.

   (f) $\mathbb{R} \sim (0, 1)$.

2. Prove theorem 2.15.

3. Show that if $A \sim C_m$ and $A \sim C_n$ then $m = n$.

4. Show that set $\mathbb{N} = \mathbb{Z}^+$ is infinite.

Definition 2.5 (Countable sets)**.** A set $\mathcal{A}$ is called **countable** (or denumerable) if $\mathcal{A} \sim \mathbb{Z}^+$ or $\mathbb{Z}^+ \sim \mathcal{A}$. A set $\mathcal{A}$ is called **at most countable** if it is finite or countable. A set which is neither **finite** nor **countable** is called **uncountable**.

Remark. This definition is not standard. Some authors call a set $\mathcal{A}$ *countable* if $\mathcal{A} \sim \mathbb{Z}^+$ or $\mathcal{A}$ is finite.

Example 2.6.

- $\mathbb{N} = \mathbb{Z}^+$ is countable since $\mathbb{Z}^+ \sim \mathbb{Z}^+$.
- $2\mathbb{Z}$ is countable. (By exercise 1b)
- $\mathbb{Z}$ is countable.

  Consider $f : \mathbb{Z} \longrightarrow \mathbb{Z}^+$ by

$$f(n) = \begin{cases} \dfrac{n}{2}, & n \text{ is even,} \\ -\dfrac{n-1}{2}, & n \text{ is odd.} \end{cases}$$

  Since this function f is bijective, it is clear that $\mathbb{Z}$ is countable.

Remark. A countable set is a set which can be listed by

$$\mathcal{A} = \{a_1, a_2, \ldots, a_n, \ldots\}$$

because of the bijective function $f : \mathbb{Z}^+ \longrightarrow \mathcal{A}$,

$$f(1) = a_1, f(2) = a_2, \ldots f(n) = a_n, \ldots$$

Recall : Let A, B be two sets, **the Cartesian product** of A and B is the set

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Theorem 2.17. *If A and B are countable sets, then their Cartesian product $A \times B$ is countable.*

Proof. As A and B are countable, we can list theirs elements as

$$\begin{aligned} A &= \{a_1, a_2, \ldots, a_m, \ldots\}, \\ B &= \{b_1, b_2, \ldots, b_n, \ldots\}, \\ \text{So } A \times B &= \{(a_m, b_n) : a_m \in A, b_n \in B\}. \end{aligned}$$

Write the elements of $A \times B$ as follows :

$$\begin{array}{l} (a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, b_4), \ldots \\ (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, b_4), \ldots \\ (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_3, b_4), \ldots \\ (a_4, b_1), (a_4, b_2), (a_4, b_3), (a_4, b_4), \ldots \end{array}$$

Now re-indices, write $A \times B = \{c_1, c_2, c_3, \ldots\}$ where c_i are chosen along the arrows

$$\begin{array}{rl} 1^{\text{st}} \text{ arrow} & c_1 = (a_1, b_1) \\ 2^{\text{nd}} \text{ arrow} & c_2 = (a_1, b_2) \\ & c_3 = (a_2, b_1) \\ 3^{\text{rd}} \text{ arrow} & c_4 = (a_1, b_3) \\ & c_5 = (a_2, b_2) \\ & c_6 = (a_3, b_1) \\ & \vdots \end{array}$$

In general, if (a_m, b_n) is an arbitrary element in $A \times B$, there exists an arrow through (a_m, b_n), the $(m+n-1)^{\text{th}}-$arrow. Along this arrow there are finitely many elements. Counting the number of elements, there exists an i so that $c_i = (a_m, b_n)$. This shows that we can list the elements of $A \times B$ by $c_1, c_2, c_3, \ldots$, i.e. $A \times B$ is countable.

Corollary 2.18. $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}^+ \times \mathbb{Z}$, $\mathbb{Z}^+ \times \mathbb{Z}^+$, $\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n-times}$ *are countable.*

Remark : Let T be a nonempty subset of $\mathbb{Z}^+$ then T contains a minimum. (Prove by the use of property of **least-upper-bound**.)

Theorem 2.19. *Let S be an at most countable set, and $T \subseteq S$, then T is at most countable.*

Proof. If S is finite, then T must be finite, this implies T is at most countable. Assume, S is infinite. If T is finite T is also clear to be at most countable. We then consider only the case T is infinite. As S is countable, there exists a bijection $f : \mathbb{Z}^+ \longrightarrow S$. Setting $s_n = f(n)$, so we can list the elements of S,

$$S = \{s_1, s_2, s_3, \ldots\}.$$

Set $I = \{n : s_n \in T\}$ then I is infinite. Let a_1 be the smallest element of I, a_2 be the smallest element of $I \backslash \{a_1\}$, a_3 be the smallest element of $I \backslash \{a_1, a_2\}$, ...

In general, suppose we have found $a_1, a_2, \ldots, a_k$, let a_k be the smallest element of $I \backslash \{a_1, a_2, \ldots, a_{k-1}\}$ Define $g : \mathbb{Z}^+ \longrightarrow A$ by $g(k) = a_k$. Clearly, g is a bijective by the construction of a_k. So $f \circ g$ is an injective map of $\mathbb{Z}^+$ into T. Consider $s_n \in T$, the $n \in I$ then we can write $a_k = n$ for some k. Thus

$$(f \circ g)(k) = f(g(k)) = f(a_k) = f(n) = s_n.$$

Also $f \circ g$ is surjective. Thus $f \circ g$ is a bijective of $\mathbb{Z}^+$ onto T. Then T is countable.

Theorem 2.20. *The rational number set $\mathbb{Q}$ is countable.*

Proof Exercise.

Theorem 2.21 (Dedekind's Theorem)**.** *A set S is infinite iff S is cardinally equivalent to a* ***proper*** *subset of itself.*

Proof. The proof is left to be an exercise. The following steps outline a proof of Dedekind's theorem

1. Every countable (denumerable) set is equivalent to a proper subset of itself.

2. Every infinite set has a countable subset.

3. Every infinite set is equivalent to a proper subset of itself.

4. A finite set is not equivalent to any of its proper subsets.

We have already seen that $2\mathbb{Z} \subseteq \mathbb{Z}$ and $2\mathbb{Z} \sim \mathbb{Z}$. Furthermore, $(0,1) \subseteq \mathbb{R}$ and $(0,1) \sim \mathbb{R}$. By the theorem, $\mathbb{Z}$ and $\mathbb{R}$ are infinite.

Exercise

1. Prove that the rational number set $\mathbb{Q}$ is countable.

2. Prove the Dedekind's theorem.

2.3 Intervals

Let a, b be real numbers so that $a < b$. We set

- $(a,b) = \{x \in \mathbb{R} : a < x < b\}$, we read "open interval from a to b".
- $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, we read "closed interval from a to b".
- $\left.\begin{array}{ll} [a,b) & = \{x \in \mathbb{R} : a \leq x < b\} \\ (a,b] & = \{x \in \mathbb{R} : a < x \leq b\} \end{array}\right\}$, we read "half open intervals from a to b".

a and b are called the end points of these intervals. We also define unbounded intervals :

- $(a,\infty) = \{x \in \mathbb{R} : a < x\}$.
- $[a,\infty) = \{x \in \mathbb{R} : a \leq x\}$.
- $(-\infty,b) = \{x \in \mathbb{R} : x < b\}$.

- $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.
- $(-\infty, \infty) = \mathbb{R}$.

The interval $(0,1)$ is called the open unit interval, sometimes written I or U.

Remark. Every real number $x \in (0,1)$ can be expressed as an unending decimal

$$0.d_1 d_2 d_3 \cdots d_k \cdots,$$

where $d_k \in \{0,1,2,\cdots,9\}$. This representation is unique except that a decimal ending in all zeroes

$$0.d_1 d_2 d_3 \cdots d_k 000 \cdots, \tag{2.3}$$

$d_k \neq 0$ is equivalent to

$$0.d_1 d_2 d_3 \cdots (d_k - 1) 999 \cdots$$

(Why?) To avoid ambiguity we will use only the form with an ending string of 0s. Conversely, every unending decimal as in (2.3) represents a real number $x \in (0,1)$.

Theorem 2.22 (Cantor's Theorem)**.** *The open unit interval* $(0,1)$ *is uncountable.*

Proof.

1. $(0,1)$ **is infinite.**

   Consider the map $f : (0,1) \longrightarrow (0, \frac{1}{2})$ defined by

   $$f(x) = \frac{x}{2}.$$

   Clearly, f is a bijective function. So $(0,1)$ is cardinally equivalent to $(0, \frac{1}{2})$. By Dedekind's Theorem $(0,1)$ is infinite.

2. $(0,1)$ **is not countable.** (Proof by contradiction.)

   Suppose to the contrary that $(0,1)$ is countable. Then we can list the elements of $(0,1)$ as $x_1, x_2, x_3, \ldots$

Let x_i be expressed by an unending decimal :

$$x_i = 0.d_{i1}d_{i2}d_{i3}\cdots d_{ik}\cdots,$$

where x_i is not ending in all 9s. So this representation is unique. Write all these elements in $(0,1)$ in decimal representations as indicated above :

$$\begin{aligned}
x_1 &= 0.d_{11}d_{12}d_{13}\cdots d_{1n}\cdots \\
x_2 &= 0.d_{21}d_{22}d_{23}\cdots d_{2n}\cdots \\
x_3 &= 0.d_{31}d_{32}d_{33}\cdots d_{3n}\cdots \\
&\vdots \\
x_n &= 0.d_{n1}d_{n2}d_{n3}\cdots d_{nn}\cdots \\
&\cdots
\end{aligned}$$

Define $y = 0.e_1e_2e_3\cdots e_n\cdots$ where

$$e_n = \begin{cases} 2 & \text{if } d_{nn} = 1, \\ 1 & \text{if } d_{nn} \neq 1, \end{cases}$$

This is clear that $y \in (0,1)$. Since the n^{th} digit of x_n and y are different ($d_{nn} \neq e_n$). Since y is different from all x_n $(n = 1, 2, \ldots)$ However $x_1, x_2, \ldots, x_n, \ldots$ comprise all of elements of $(0,1)$ and we see that $y \neq x_i$. This means $y \notin \{x_1, x_2, \ldots\} = (0,1)$ which contradicts to the fact that $y \in (0,1)$. Hence $(0,1)$ is not countable.

By all of above, $(0,1)$ is **uncountable**.

Theorem 2.23. *Suppose A and B are countable then $A \cup B$ is countable.*

Proof. By the countability of A and B, there exists bijections $f : A \longrightarrow \mathbb{Z}^+$ and $g : B \longrightarrow \mathbb{Z}^+$. Define $h : A \cup B \longrightarrow \mathbb{Z}^+$ by

$$h(x) = \begin{cases} 2f(x) - 1, & \text{if } x \in A, \\ 2g(x), & \text{if } x \in B, x \notin A. \end{cases}$$

Then h is an injection from $A \cup B$ into $\mathbb{Z}^+$. Now $A \cup B$ and $h(A \cup B)$ are cardinally equivalent. By theorem 2.19, $h(A \cup B)$ is at most countable and by the cardinality $A \cup B$ must be countable. However, $A \subseteq A \cup B$ and A is infinite $\Rightarrow A \cup B$ is infinite. Then $A \cup B$ is countable.

Remark. Theorem 2.23 holds also if the word "**countable**" is replaced by "**at most countable**".

Let $\{A_1, A_2, \ldots, A_n, \ldots\}$ be an infinite collection of countable sets. Let

$$S = \bigcup_{n=1}^{\infty} A_n$$

be defined by

$$x \in S \Leftrightarrow x \in A_n \text{ for at least one } n.$$

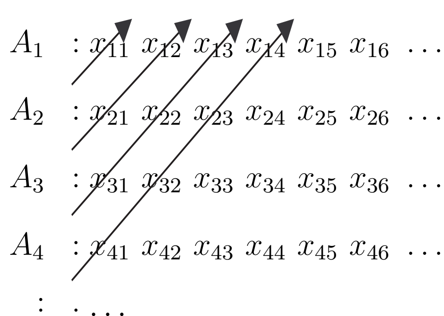
Theorem 2.24. *Let $\{A_1, A_2, \ldots, A_n, \ldots\}$ be a countable collection of countable sets. Then $\bigcup_{n=1}^{\infty} A_n$ is countable.*

Proof. (Compare to the proof of theorem 2.17.)

As each A_n is countable, we can list its elements

$$A_n = \{x_{n1}, x_{n2}, x_{n3}, \ldots, x_{nk}, \ldots\}$$

List all elements in each set as follow :



Now we list all the elements of $\bigcup_{n=1}^{\infty} A_n$ by moving along the arrow beginning in the top-left corner. It may happen some x_{ij} appear move that once, i.e. we may have $x \in A_n$ and $x \in A_m$ for $m \neq n$. In this case, we list this x as it appears for the first time only. So $\bigcup_{n=1}^{\infty} A_n$ is a subset of the following list

$$x_{11}, x_{21}, x_{12}, x_{13}, x_{22}, x_{31}, \ldots$$

By theorem 2.19, $\bigcup_{n=1}^{\infty} A_n$ is at most countable. However, $A_n \in \bigcup_{n=1}^{\infty} A_n$ and A_n is infinite. Then $\bigcup_{n=1}^{\infty} A_n$ must be countable.

Exercise

1. Prove that closed interval $[0, 1]$ is uncountable.

2. Prove that $\mathbb{R}$ is uncountable.

3. Prove that the set of irrational numbers is uncountable.

4. Prove that the unit square $I \times I = \{(x, y) : 0 < x, y < 1\}$ is uncountable.

Chapter 3

Sequences and Series

3.1 Sequences

3.1.1 Absolute Values

Definition 3.1 (Absolute Values)**.** Let $x \in \mathbb{R}$. The absolute value of x is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Theorem 3.1 (Properties of the absolute value)**.** *Let* $x, y \in \mathbb{R}, a > 0$

1. $|x| \geq 0$
2. $|x| = 0 \iff x = 0$
3. $|x||y| = |xy|$
4. $-|x| \leq x \leq |x|$
5. $|x| \leq a \iff -a \leq x \leq a$
6. $|x| < a \iff -a < x < a$

7. $|x| \geq a \Longleftrightarrow x \leq -a$ *or* $x \geq a$

8. $|x| > a \Longleftrightarrow x < -a$ *or* $x > a$

9. $|x + y| \leq |x| + |y|$ *(triangle inequality)*

10. $||x| - |y|| \leq |x - y|$ *(second triangle inequality)*

Proof. Item 1) - 3) : exercise

4. **Case I** $x \geq 0$

   Then $|x| = x$. So $-|x| \leq x = |x|$. Thus $-|x| \leq x \leq |x|$.

   Case II $x < 0$

   Then $|x| = -x$. So $-|x| = x \leq |x|$. Hence $-|x| \leq x \leq |x|$.

   By the both cases, $-|x| \leq x \leq |x|$.

5. ($\Rightarrow$) Suppose that $|x| \leq a$ then $-|x| \geq -a$. By 4), $-a \leq -|x| \leq x \leq |x| \leq a$

   ($\Leftarrow$) Suppose $-a \leq x \leq a$

   (a) $x \geq 0$ then $|x| = x$, then $-a \leq |x| = x \leq a \Rightarrow |x| \leq a$.

   (b) $x < 0$ then $|x| = -x$, then $-a \leq -|x| = x \leq a \Rightarrow -a \leq -|x| \Rightarrow a \geq |x|$.

   By both cases, $|x| \leq a$.

6. Similar to the proof of 5).

7. **Case I** $x > 0$

   Then $x = |x| > a$

   Case II $x < 0 \Rightarrow |x| = -x$

   Then $-x = |x| > a \Rightarrow x < -a$

   By both cases, $|x| > a \Rightarrow x > a$ or $x < -a$.

8. Similar to the proof of 7).

9. $|x+y| \leq |x| + |y|$ (triangle inequality) By 4.), we have

$$-|x| \leq x \leq |x|$$
$$-|y| \leq y \leq |y|$$
$$\Rightarrow -(|x| + |y|) \leq x + y \leq |x| + |y|$$

   By the property 5.) (with $a = |x| + |y|$) we have

$$|x+y| \leq |x| + |y|.$$

10. $||x| - |y|| \leq |x - y|$ (Exercise)

Exercise

1. Prove theorem 3.1 for the items 1-3 and the second triangle inequality.

2. Draw the graph of function $f(x) = \dfrac{|x|}{x}$, where $x \in \mathbb{R}$ and $x \neq 0$.

3.1.2 Sequences

Definition 3.2 (Sequences)**.** A **sequence** is a function whose domain is the set of positive integers $\mathbb{Z}^+$ (or set of natural numbers $\mathbb{N}$)

Remark.

1. If s is a sequence, we set

$$s_n = s(n), \qquad n \in \mathbb{N}.$$

   s_n is called the n**-th term** of the sequence, and n is also called the **index** of the sequence.

We can write a sequence by the listing the elements of its range ordered by index :

$$s = \{s_1, s_2, s_3, \ldots\} \quad \text{or} \quad s = \{s_n\}_{n=1}^{\infty} \quad \text{or} \quad s = \{s_n\}.$$

Indeed, there are many other ways to write a term of sequence.

2. Even the definition of sequence is defined on a national number set $\mathbb{N}$, it may sometimes define a sequence starting from some other numbers different from 1, e.g. $\left\{\frac{1}{n+4}\right\}_{n=-3}^{\infty}$, $\{(-1)^n\}_{n=0}^{\infty}$, $\{(n-8)^2\}_{n=5}^{\infty}$, etc.

Example 3.1. These are examples of writing sequences :

- The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ can be also written as $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$
- The sequence $\{2n\}_{n=1}^{\infty}$ can be also written as $\{2, 4, 6, \ldots, 2n, \ldots\}$ or $s_1 = 2, s_2 = 4, s_3 = 6, \ldots, s_n = 2n, \ldots$
- $\{(-1)^n + 1\}_{n=1}^{\infty}$ We can write it as $\{0, 2, 0, 2, 0, \ldots\}$
- $\{(1 + 1/n)^n\}_{n=1}^{\infty}$ It is equivalent to $\{2, 2.25, 2.37\dot{0}3\dot{7}, \ldots\}$
- $\left\{a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}\right\}_{n=1}^{\infty}$ where $a_1 = 1$. This sequence is defined recurrently. It may be written by $\{1, 1.5, 1.41\dot{6}, 1.414215..., \ldots\}$

3.1.3 Monotonicity

Definition 3.3 (Monotonicity)**.** Let $s = \{s_n\}$ be a sequence. We say that

- s is **increasing** if $s_n \leq s_{n+1}, \forall n \in \mathbb{N}$
- s is **strictly increasing** if $s_n < s_{n+1}, \forall n \in \mathbb{N}$
- s is **decreasing** if $s_n \geq s_{n+1}, \forall n \in \mathbb{N}$

- s is **strictly decreasing** if $s_n > s_{n+1}, \forall n \in \mathbb{N}$

Any sequence with any of these properties is called **monotone**. By induction we can define definition 3.3 by

Definition 3.4 (Monotonicity (alternative definition))**.** Let $s = \{s_n\}$ be a sequence. We say that

- s is **increasing** if $s_n \leq s_m, \forall m > n \in \mathbb{N}$
- s is **strictly increasing** if $s_n < s_m, \forall m > n \in \mathbb{N}$
- s is **decreasing** if $s_n \geq s_m, \forall m > n \in \mathbb{N}$
- s is **strictly decreasing** if $s_n > s_m, \forall m > n \in \mathbb{N}$

Why? (exercise)

Example 3.2. Consider the following statements :

- Is the sequence $s = \{\frac{1}{n}\}$ monotone?

  Since $s = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, it appears that s is decreasing. Consider

$$s_{n+1} - s_n = \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} < 0, \forall n \in \mathbb{N}$$

  So $s_{n+1} < s_n \Rightarrow s$ is **strictly decreasing**.

- Is the sequence $\{2^n\}$ monotone?

Method I $s_{n+1} - s_n = 2^{n+1} - 2^n = 2^n(2-1) = 2^n > 0 \Rightarrow s_{n+1} > s_n$ then s is strictly increasing.

Method II $\dfrac{s_{n+1}}{s_n} = \dfrac{2^{n+1}}{2^n} = 2 > 1 \Rightarrow s_{n+1} > s_n$ strictly increasing.

- Is the sequence $\left\{\dfrac{n^2}{n^2-1}\right\}_{n=2}^{\infty}$ monotone?

  The sequence can be listed by

$$\frac{4}{3}, \frac{9}{8}, \frac{16}{15}, \frac{25}{24}, \ldots$$

  Since $s_n = \dfrac{n^2}{n^2-1} = \dfrac{n^2-1+1}{n^2-1} = 1 + \dfrac{1}{n^2-1}$ then

$$\begin{aligned}
s_{n+1} - s_n &= \left(1 + \frac{1}{(n+1)^2-1}\right) - \left(1 + \frac{1}{n^2-1}\right) \\
&= \frac{1}{(n+1)^2-1} - \frac{1}{n^2-1} \\
&= \frac{(n^2-1) - (n^2+2n)}{((n+1)^2-1))\,(n^2-1))} \\
&= \frac{-1-2n}{((n+1)^2-1))\,(n^2-1))} < 0 \\
\Rightarrow \quad & s_{n+1} < s_n
\end{aligned}$$

  This sequence is strictly decreasing.

- The constant sequence $s = \{c\}_{n=1}^{\infty}$ (c is a constant) has elements

$$c, c, c, \ldots$$

  This sequence is both increasing and decreasing.

- Consider the sequence $\left\{\left\lfloor\dfrac{n}{2}\right\rfloor\right\}_{n=1}^{\infty}$. It this sequence monotone?

  List the elements of this sequence :

$$0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$$

  It appears that the sequence is increasing.

Definition 3.5 (Bounded sequences)**.** Let $s = \{s_n\}_{n=1}^{\infty}$ be a sequence. We say that

- s is **bounded above** if there exists a real number M so that $s_n \leq M \forall n \in \mathbb{N}$.

- s is **bounded below** if there exists a real number m so that $s_n \geq m \forall n \in \mathbb{N}$.
- s is **bounded** if it is bounded above **and** bounded below.
- s is **unbounded** if it is not bounded.

Remark. These definitions are not really new. We have discussed the definition of boundedness of sets in definition 2.1 (page 35).

Theorem 3.2. *A sequence $\{s_n\}$ is bounded if and only if there exists a real number M such that*

$$|s_n| \leq M, \qquad \forall n \in \mathbb{N}.$$

Proof. Exercise.

Example 3.3. Consider the following sequences

- The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is bounded since $0 < \frac{1}{n} < 1 \ \forall n \in \mathbb{N}$.
- $\{n^2\}_{n=1}^{\infty}$ This sequence is bounded below, as $0 < n^2 \ \forall n$ but **not** bounded above.
- $\{(-1)^n n^2\}_{n=1}^{\infty}$ This sequence is not rather bounded below nor bounded above. It is unbounded.

Exercise

1. Consider the following sequences. Determine that whether they are increasing, strictly increasing, decreasing, strictly decreasing, bounded or not.
   (a) $\{2^n - n^2\}_{n=1}^{\infty}$
   (b) $\{2^n - n^2\}_{n=4}^{\infty}$
   (c) $\left\{\frac{n^2-1}{n^2}\right\}_{n=1}^{\infty}$
   (d) $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$
   (e) $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$
   (f) $\left\{\frac{2^n}{n}\right\}_{n=1}^{\infty}$
   (g) $\{\sin^2(n) + \cos^2(n)\}_{n=1}^{\infty}$

2. Prove theorem 3.2

3. If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are sequences, then the sum $s + t$ is the sequence whose n-th term is $s_n + t_n$. That is

$$s + t = \{s_n + t_n\}_{n=1}^{\infty}$$

Similarly, the difference, product and quotient are given by

$$\begin{aligned} s - t &= \{s_n - t_n\}_{n=1}^{\infty} \\ st &= \{s_n t_n\}_{n=1}^{\infty} \\ \frac{s}{t} &= \left\{\frac{s_n}{t_n}\right\}_{n=1}^{\infty} \qquad \text{provided } t_n \neq 0 \text{ for } n \in \mathbb{Z}^+ \end{aligned}$$

Prove or disprove the following statements.

(a) if s and t are both increasing, then so is their sum.

(b) if s and t are both increasing, then so is their difference.

(c) if s and t are both increasing, then so is their product.

(d) if s and t are both increasing, then so is their quotient.

(e) if s and t are both increasing sequences of positive numbers, then so is their product.

(f) if s and t are both increasing sequences of positive numbers, then so is their quotient.

(g) if s and t are both bounded, then so is their difference.

(h) if s and t are both bounded sequences of positive numbers, then so is their product.

(i) if s and t are both bounded sequences of positive numbers, then so is their quotient.

3.2 Subsequences

Suppose, we are given a sequence $s = \{s_n\}_{n=1}^{\infty}$ whose elements are listed as follows:

$$s_1, s_2, s_3, \ldots, s_n, \ldots$$

We now pick a subset of this set by choosing values for the index :

$$n_1 < n_2 < n_3 < \cdots < n_k < \cdots$$

Define a sequence $\{t_k\}_{k=1}^{\infty}$ by $t_k = s_{n_k}$. We also write this sequence as $\{s_{n_k}\}_{k=1}^{\infty}$ and call it a **subsequence** of $s = \{s_n\}_{n=1}^{\infty}$

Example 3.4. Let $s = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ which can be written as

$$s = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

1. Choose $n_1 = 2, n_2 = 4, n_3 = 6, \ldots, n_k = 2k, \ldots$ and we get the subsequence

$$\{t_k\}_{k=1}^{\infty} = \{s_{2k}\}_{k=1}^{\infty} = \left\{\frac{1}{2k}\right\}_{k=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right\}$$

2. Another sequence is $n_1 = 5, n_2 = 6, n_3 = 7, \ldots, n_k = k+4, \ldots$ This sequence is obtained by removing the first few elements of s. It is called the tail of s.

$$\{t_k\}_{k=1}^{\infty} = \{s_{k+4}\}_{k=1}^{\infty} = \left\{\frac{1}{k+4}\right\}_{k=1}^{\infty} = \{s_n\}_{n=5}^{\infty} = \left\{\frac{1}{n}\right\}_{n=5}^{\infty} = \left\{\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots\right\}$$

Remark. In general, if $s = \{s_n\}_{n=1}^{\infty}$ is a sequence and if $p \in \mathbb{N}$ is any natural number then $t = \{s_n\}_{n=p}^{\infty}$ is called the **tail** of $s = \{s_n\}_{n=1}^{\infty}$.

3. Let $s = \{n + (-1)^n n\}_{n=1}^{\infty} = \{0, 4, 0, 6, 0, 8, \ldots\}$. Consider

$$t = \{s_{2k-1}\}_{k=1}^{\infty} = \left\{(2k-1) + (-1)^{2k-1}(2k-1)\right\}_{k=1}^{\infty} = \{0\}_{k=1}^{\infty}$$

This subsequence t is bounded even sequence s is unbounded.

4. Let $s = \{(-1)^n n\}_{n=1}^{\infty} = \{-1, 2, -3, 4, -5, 6, \ldots\}$. Consider

$$t = \{s_{2k}\}_{k=1}^{\infty} = \left\{(-1)^{2k}(2k)\right\}_{k=1}^{\infty} = \{2k\}_{k=1}^{\infty} = \{2, 4, 6, 8, \ldots\}$$

This subsequence t is strictly increasing where sequence s is not.

Exercise

1. Let s be the set of real numbers. Prove or disprove the following statements.

   (a) If s is finite then s contains a largest element (which is the maximum of s).

   (b) If s is infinite and bounded then s need not to have a largest element.

2. Given $\{s_n\}_{n=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$, find a formula for the k-th term of the subsequence $t = \{s_{n_k}\}_{k=1}^{\infty}$, and compute t_1, t_2 and t_3.

   (a) $s_n = \dfrac{1}{n}$; $n_k = 4k + 3$

   (b) $s_n = (-1)^{n+1} n$; $n_k = 3k$

   (c) $s_n = \dfrac{(-1)^{n+1}}{n^2}$; $n_k = 2k - 1$

   (d) $s_n = \begin{cases} n^2 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$; $n_k = 2k$

   (e) $s_n = \begin{cases} n^2 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$; $n_k = 2k + 1$

3. Prove the following statements

   (a) A subsequence of an increasing sequence is increasing.

   (b) A subsequence of a bounded sequence is bounded.

   (c) If t is a subsequence of s and u is a subsequence of t, then u is a subsequence of s.

3.3 Null sequences

Consider the sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2}, \ldots$ which can be written as

$$s = \left\{ \frac{1}{n^2} \right\}_{n=1}^{\infty}.$$

As n gets larger and larger, the term $\frac{1}{n^2}$ of the sequence come closer and closer to 0.

Example 3.5. Consider sequence $s = \left\{ s_n = \frac{1}{n^2} \right\}_{n=1}^{\infty}$.

- Find all terms s_n which $s_n = \frac{1}{n^2} < \frac{1}{100}$: Consider

$$\begin{aligned} \frac{1}{n^2} &< \frac{1}{100} \\ n^2 &> 100 \\ n &> 10 \end{aligned}$$

  That is if $n = 11, 12, 13, \ldots$ then $\frac{1}{n^2} < \frac{1}{100}$.

- Find all terms s_n which $\frac{1}{n^2} < 0.0001$: Consider

$$\begin{aligned} \frac{1}{n^2} &< 0.0001 \\ n^2 &> \frac{1}{0.0001} = 10000 \\ n &> 100 \end{aligned}$$

  That is if $n = 101, 102, 103, \ldots$ then $\frac{1}{n^2} < 0.0001$.

- Given $\varepsilon > 0$, find all terms s_n which $\frac{1}{n^2} < \varepsilon$:

$$\begin{aligned} \frac{1}{n^2} &< \varepsilon \\ n^2 &> \frac{1}{\varepsilon} \\ n &> \sqrt{\frac{1}{\varepsilon}}. \end{aligned}$$

Applying corollary 2.11 (page 44) shows that there must exist $N \in \mathbb{N}$ so that $N > \sqrt{\frac{1}{\varepsilon}}$. So if $n \geq N$, we have

$$\frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

Definition 3.6 (Null sequences)**.** A sequence $\{s_n\}_{n=1}^{\infty}$ is **null** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $|s_n| < \varepsilon$ for all $n \geq N$.

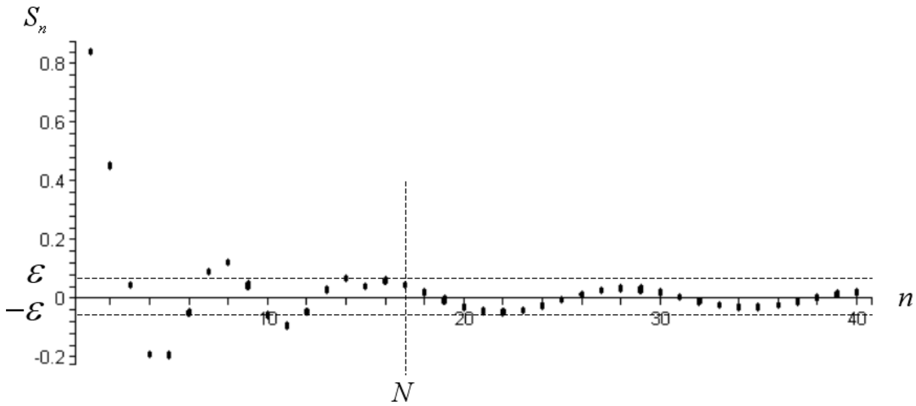


Figure 3.1: When $n > N$, $|s_n| < \varepsilon$

Example 3.6. These are examples of null sequences and not null sequence.

1. $\left\{\frac{(-1)^n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is null.

   Here $s_n = \frac{(-1)^n}{\sqrt{n}}$. Given $\varepsilon > 0$, we must find $N \in \mathbb{N}$ so that $|s_n| < \varepsilon$ for $n \geq N$.

$$\begin{aligned} |s_n| < \varepsilon \quad \Rightarrow \quad \left|\frac{(-1)^n}{\sqrt{n}}\right| &< \varepsilon \\ \frac{1}{\sqrt{n}} &< \varepsilon \\ \sqrt{n} &> \frac{1}{\varepsilon} \\ n &> \frac{1}{\varepsilon^2} \end{aligned}$$

   So pick $N \in \mathbb{N}$ so that $N > \frac{1}{\varepsilon^2}$ (By Archimedian property) then if $n \geq N$ we have $n \geq N > \frac{1}{\varepsilon^2}$ so that going backward

$$\left|\frac{(-1)^n}{\sqrt{n}}\right| < \varepsilon$$

thus $\left\{\dfrac{(-1)^n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is a null sequence.

2. The sequence $\left\{\dfrac{3n^2-2}{2n^3+4n}\right\}_{n=1}^{\infty}$ is a null sequence.

Proof.

$$\begin{aligned}\left|\frac{3n^2-2}{2n^3+4n}\right| &= \frac{3n^2-2}{2n^3+4n}, \quad \forall n = 1,2,3,\dots \\ &< \frac{3n^2}{2n^3+4n} \\ &< \frac{3n^2}{2n^3} = \frac{3}{2n} \qquad (3.1)\end{aligned}$$

Given $\varepsilon > 0$, we want $|s_n| < \varepsilon$. By equation (3.1), it is enough to make $\dfrac{3}{2n} < \varepsilon \Rightarrow n > \dfrac{3}{2\varepsilon}$. Choose $N < \dfrac{3}{2\varepsilon}$ so if $n \geq N$ then by equation (3.1)

$$|s_n| < \frac{3}{2n} \leq \frac{3}{2N} \leq \frac{3}{2\left(\frac{3}{2\varepsilon}\right)} = \varepsilon.$$

This shows that $\{s_n\}$ is a null sequence.

3. Consider the sequence $\left\{\dfrac{n^2+1}{2n^3-n}\right\}_{n=1}^{\infty}$

Claim : This is a null sequence.

Proof :

$$|s_n| = \left|\frac{n^2+1}{2n^3-n}\right| = \frac{n^2+1}{2n^3-n} \leq \frac{n^2+n^2}{2n^3-n} = \frac{2n^2}{2n^3-n} \leq \frac{2n^2}{2n^3-n^3} = \frac{2}{n} \qquad (3.2)$$

Given $\varepsilon > 0$, we want $|s_n| < \varepsilon$. By equation (3.2) it is enough to make $\dfrac{2}{n} < \varepsilon \Rightarrow n > \dfrac{2}{\varepsilon}$.

Choose $N \in \mathbb{N}$ with $N > \dfrac{2}{\varepsilon}$. By equation (3.2) if $n \geq N \Rightarrow n \geq N > \dfrac{2}{\varepsilon}$ so that $|s_n| < \dfrac{2}{n} < \dfrac{2}{\frac{2}{\varepsilon}} = \varepsilon$ thus the sequence is null.

4. Consider the sequence $\left\{\dfrac{n^2+1}{n^2+4n}\right\}_{n=1}^{\infty}$

Claim : This is **not** a null sequence.

Proof :

$$|s_n| = \left|\frac{n^2+1}{n^2+4n}\right| = \frac{n^2}{n^2+4n} \geq \frac{n^2}{n^2+4n^2} = \frac{n^2}{5n^2} = \frac{1}{5}, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Choose $\varepsilon = \frac{1}{10}$ then $|s_n| > \frac{1}{5} > \frac{1}{10} = \varepsilon$. So we can find no n with $|s_n| < \varepsilon$. Thus $\{s_n\}$ is not a null sequence.

Theorem 3.3. *If $\{s_n\}_{n=1}^{\infty}$ is a null sequence, then $\{-s_n\}_{n=1}^{\infty}$ is also a null sequence.*

Proof. Given $\varepsilon > 0$ then there must exists $N \in \mathbb{N}$ so that

$$|s_n| < \varepsilon, \qquad n \geq N.$$

Also $|-s_n| = |s_n| < \varepsilon, n \geq N$. This shows that $\{-s_n\}_{n=1}^{\infty}$ is also a null sequence.

Theorem 3.4. *If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are null sequences, then $\{s_n + t_n\}_{n=1}^{\infty}$ is also a null sequence.*

Proof.

- **Discussion :** We want $|s_n + t_n| < \varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. This holds surely :

$$|s_n| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |t_n| < \frac{\varepsilon}{2}.$$

- **Real proof :** Let $\varepsilon > 0$ be given.

  Because $\{s_n\}$ is null, there exists $N_s \in \mathbb{N}$ so that $|s_n| < \frac{\varepsilon}{2}$, $\forall n \geq N_s$.

  Similarly, as $\{t_n\}$ is null, there exists $N_t \in \mathbb{N}$ so that $|s_n| < \frac{\varepsilon}{2}$, $\forall n \geq N_t$.

  Set $N = \max(N_s, N_t)$ then if $n \geq N$

$$|s_n + t_n| \leq |s_n| + |t_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Example 3.7. $\left\{\dfrac{12n^4 - 2n^2 + 6}{n(3n^2+4)(3n^2-1)}\right\}_{n=1}^{\infty}$

Since $\dfrac{12n^4 - 2n^2 + 6}{n(3n^2+4)(3n^2-1)} = \dfrac{3n^2-2}{3n^3+4n} + \dfrac{n^2+1}{3n^3-n}$ and by examples 3.6.2 and 3.6.3 which show that $\left\{\dfrac{3n^2-2}{3n^3+4n}\right\}$ and $\left\{\dfrac{n^2+1}{3n^3-n}\right\}$ are null, then $\left\{\dfrac{12n^4 - 2n^2 + 6}{n(3n^2+4)(3n^2-1)}\right\}_{n=1}^{\infty}$ is null.

Corollary 3.5. *If* $\{s_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *are null sequences, then* $\{s_n - t_n\}_{n=1}^{\infty}$ *is also a null sequence.*

Proof. Exercise.

Theorem 3.6. *If* $\{s_n\}_{n=1}^{\infty}$ *is bounded and* $\{t_n\}_{n=1}^{\infty}$ *is a null sequence, then* $\{s_n t_n\}_{n=1}^{\infty}$ *is also a null sequence.*

Proof.

- **Discussion :** We want : $|s_n t_n| < \varepsilon \Rightarrow |s_n||t_n| < \varepsilon$. This is true if $|s_n| < M$ and $|t_n| < \dfrac{\varepsilon}{M}$

- **Real Proof :** Let $\varepsilon > 0$ be given. Because $\{s_n\}$ is bounded, then there exists $M > 0$ so that $|s_n| < M$ for $n = 1, 2, 3, \ldots$

  Because $\{t_n\}$ is a null sequence (with $\dfrac{\varepsilon}{M}$ instead of ε), then there exists $N \in \mathbb{N}$ so that

  $$|t_n| < \frac{\varepsilon}{M}, \qquad \forall n \geq N.$$

  Thus

  $$|s_n t_n| = |s_n||t_n| < M\frac{\varepsilon}{M} = \varepsilon, \qquad \forall n \geq N.$$

  $\Rightarrow \{s_n t_n\}$ is a null sequence.

Example 3.8. Consider sequence $\left\{\dfrac{\sin n}{n^2}\right\}_{n=1}^{\infty}$.

$\dfrac{\sin n}{n^2} = \sin n \cdot \dfrac{1}{n^2}$. Since $|\sin n| < 1, \forall n$ then $\{\sin n\}$ is a bounded sequence and we have already seen $\left\{\dfrac{1}{n^2}\right\}$ is null, then by the theorem $\left\{\dfrac{\sin n}{n^2}\right\}$ is null.

Corollory 3.7. *If* $\{t_n\}_{n=1}^{\infty}$ *is null and* $\{c\}_{n=1}^{\infty}$ *is a constant sequence, then* $\{ct_n\}_{n=1}^{\infty}$ *is also a null sequence.*

Proof. Exercise.

Theorem 3.8. *Every null sequence is bounded.*

Proof. Let $\{s_n\}_{n=1}^{\infty}$ be null. Choose $\varepsilon = 1$. As $\{s_n\}$ is null, $\exists N \in \mathbb{N}$ so that $|s_n| < 1, \forall n \geq N$. Set $M = \max(|s_1|, |s_2|, \ldots, |s_{N-1}|, 1)$ then $|s_n| \leq M, \forall n$. Thus by theorem 3.2, $\{s_n\}_{n=1}^{\infty}$ is bounded.

Corollary 3.9. *If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are null, then $\{s_n t_n\}_{n=1}^{\infty}$ is also a null sequence.*

Proof. Exercise.

Theorem 3.10 (Squeeze Theorem)**.** *If $\{r_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are null and $\{s_n\}_{n=1}^{\infty}$ is a sequence with $r_n \leq s_n \leq t_n, \forall n$ then $\{s_n\}_{n=1}^{\infty}$ is also null.*

Proof. Let $\varepsilon > 0$ be given. Since $\{t_n\}$ is null, $\exists N_t \in \mathbb{N}$ so that

$$s_n \leq t_n \leq |t_n| < \varepsilon \qquad \forall n \geq N_t \tag{3.4}$$

Since $\{r_n\}$ is null, by theorem 3.3 $\{-r_n\}$ is also null, then $\exists N_r \in \mathbb{N}$ so that

$$-s_n \leq r_n \leq |-r_n| = |r_n| < \varepsilon \qquad \forall n \geq N_r \tag{3.5}$$

Since $|s_n| = s_n$ or $|s_n| = -s_n$, by combining inequality (3.4) and (3.5)

$$|s_n| < \varepsilon, \forall n \geq N,$$

where $N = \max(N_t, N_r)$. Hence $\{s_n\}$ is null.

Exercise

1. Prove corollary 3.5.
2. Prove corollary 3.7.
3. Prove corollary 3.9.
4. Prove that $\{s_n\}_{n=1}^{\infty}$ is null if and only if $\{|s_n|\}_{n=1}^{\infty}$ is null.
5. Prove that if $k > 0$ is a constant then $\left\{\dfrac{1}{n^k}\right\}_{n=1}^{\infty}$ is null.
6. Let $\{c\}_{n=1}^{\infty}$ be a constant sequence then $\{c\}_{n=1}^{\infty}$ is null if and only if $c = 0$.

Theorem 3.11 (Bernoulli's inequality)**.** *If $s = 1 + p$ for some $p > 0$, then $s^n \geq 1 + np$ for $n \in \mathbb{Z}^+$.*

Proof. Exercise.

Definition 3.7 (Geometric Sequences)**.** If r is any real number, then the sequence

$$\{r^n\}_{n=1}^{\infty}$$

is called the **geometric sequence**.

Theorem 3.12. *If $|r| < 1$, then the geometric sequence $\{r^n\}_{n=1}^{\infty}$ is null.*

Proof.

- **Case I :** $0 < r < 1 \Rightarrow$ then $1 < \frac{1}{r}$. By Bernoulli's inequality, if we write $\frac{1}{r} = 1 + p$ then

$$\begin{aligned} \left(\frac{1}{r}\right)^n = \frac{1}{r^n} &\geq 1 + np \\ \Rightarrow \qquad r^n &\leq \frac{1}{1+np} \\ &< \frac{1}{np} = \frac{1}{p} \cdot \frac{1}{n} \end{aligned}$$

  We have $0 < r^n < \frac{1}{p} \cdot \frac{1}{n}$. Since $\left\{\frac{1}{p}\right\}$ is a constant sequence which is bounded and $\left\{\frac{1}{n}\right\}$ is a null sequence (exercise 5., page74), by theorem 3.6, $\left\{\frac{1}{p} \cdot \frac{1}{n}\right\}$ is null. Since $\{0\}$ and $\left\{\frac{1}{p} \cdot \frac{1}{n}\right\}$ are null, by the squeeze theorem, $\{r^n\}_{n=1}^{\infty}$ is also null.

- **Case II :** $-1 < r < 0 \Rightarrow$ then $0 < -r = |r| < 1$. By case I., $\{|r|^n\} = \{|r^n|\}$ is null. By exercise 4. (page 74), $\{r^n\}$ is null.

- **Case III:** $r = 0$. Clearly $\{r^n\} = \{0\}$ is null.

3.4 Convergent sequences

Recall null sequence :

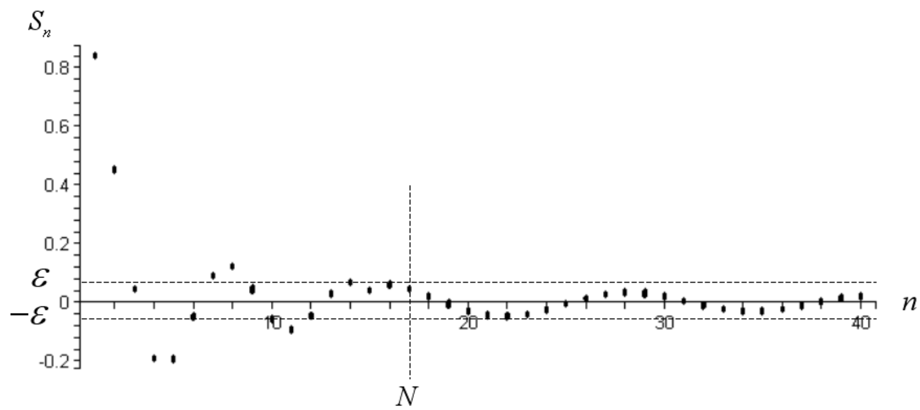


Figure 3.2: Null sequence

The terms of the above sequence approach to zero.

We may also encounter the following situation :

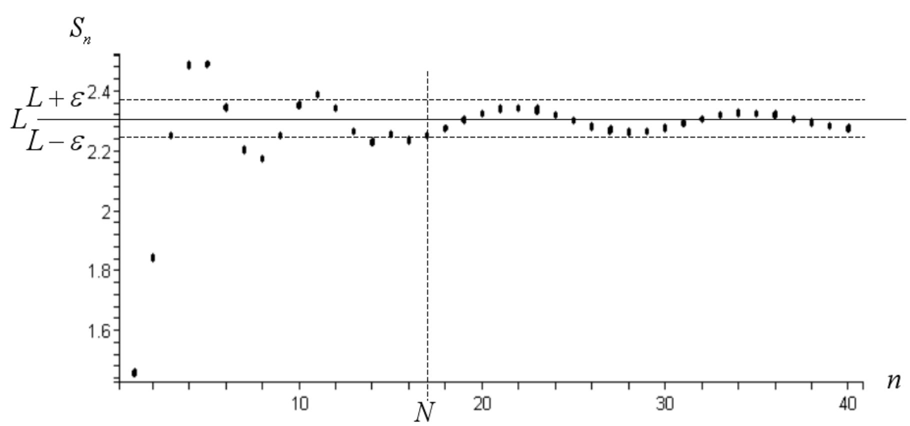


Figure 3.3: Convergent sequence

The terms of the sequence approach to L. If we subtract L from each term, we obtain a null sequence $\{s_n - L\}$.

Definition 3.8 (Convergent Sequences)**.** We say that a sequence $\{s_n\}_{n=1}^{\infty}$ converges to the number L provided that the sequence $\{s_n - L\}_{n=1}^{\infty}$ is null, we write

$$s_n \to L \qquad \text{or} \qquad \lim_{n\to\infty} s_n = L.$$

The number L is called the **limit** of the sequence. If there exists a number L so that $s_n \to L$, then we say **the sequence is convergent** (or the sequence converges). Otherwise we say that **the sequence is divergent** (or the sequence diverges).

Example 3.9. Consider the following sequences :

- the constant sequence $\{c\}_{n=1}^{\infty}$ is convergent. In fact, $\{c - c\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$ is null. Thus the constant sequence $\{c\}$ converges to c.

- $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

  For very large n, $n \approx n+1 \Rightarrow \dfrac{n+1}{n} \approx \dfrac{n}{n} = 1$. It looks like $\dfrac{n}{n+1} \to 1$.

  Claim : $\left\{\dfrac{n}{n+1} - 1\right\}_{n=1}^{\infty}$ is null.

  Proof. Now $\dfrac{n}{n+1} - 1 = \dfrac{n-(n+1)}{n+1} = \dfrac{-1}{n+1}$.

  Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that $N > \dfrac{1}{\varepsilon}$. If $n \geq N$ then

  $$\left|\frac{-1}{n+1}\right| = \frac{1}{n+1} \leq \frac{1}{N+1} < \varepsilon.$$

  This shows that $\left\{\dfrac{n}{n+1} - 1\right\}_{n=1}^{\infty}$ is null. That is $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ converges to 1.

- $\left\{\dfrac{1}{n^2}\right\}_{n=1}^{\infty}$

  By example 3.5 (page 69), we have seen that $\left\{\dfrac{1}{n^2}\right\}_{n=1}^{\infty}$ converges to 0.

By definition 3.8, if the sequence converges to the number $L = 0$, we get :

"The sequence $\{s_n\}$ converges to 0 (or $\lim_{n\to\infty} s_n = 0$) provided that $\{s_n\}$ is null."

Also by definition of null sequence 3.6 (page 70)

"A sequence $\{s_n\}_{n=1}^{\infty}$ is **null** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $|s_n| < \varepsilon$ for all $n \geq N$."

Combining the two above statements we get :

Definition 3.9 (Alternative Definition of Convergent Sequences, The $\varepsilon - N$ Formulation)**.** $\lim_{n\to\infty} s_n = L$ (or $s_n \to L$) if and only if, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_n - L| < \varepsilon$ wherever $n \geq N$.

Example 3.10. $\lim_{n\to\infty} \frac{3n^2+n+4}{6n^2} = \frac{1}{2}$

Proof. Let $\varepsilon > 0$ be given. We must find $N \in \mathbb{N}$ so that

$$\left|\frac{3n^2+n+4}{6n^2} - \frac{1}{2}\right| < \varepsilon \qquad \forall n \geq N.$$

Now

$$\left|\frac{3n^2+n+4}{6n^2} - \frac{1}{2}\right| = \left|\frac{3n^2+n+4-3n^2}{6n^2}\right| = \left|\frac{n+4}{6n^2}\right| = \frac{n+4}{6n^2} \leq \frac{n+4n}{6n^2} = \frac{5n}{6n^2} = \frac{5}{6n} \tag{3.6}$$

Pick N such that $\frac{5}{6N} < \varepsilon \Rightarrow \frac{6N}{5} > \frac{1}{\varepsilon} \Rightarrow N > \frac{5}{6\varepsilon}$

So if $n \geq N$ then, by inequality 3.6,

$$\left|\frac{3n^2+n+4}{6n^2} - \frac{1}{2}\right| < \frac{5}{6n} \leq \frac{5}{6N} < \varepsilon.$$

Theorem 3.13. *The limit of a convergent sequence is unique.*

Proof. Suppose $\{s_n\}_{n=1}^{\infty}$ is a sequence and $s_n \to L_1$ and also $s_n \to L_2$.

This means that $\{s_n - L_1\}_{n=1}^{\infty}$ and $\{s_n - L_2\}_{n=1}^{\infty}$ are null. By corollary 3.5 (page 73) the difference of sequence

$$\{(s_n - L_1) - (s_n - L_2)\}_{n=1}^{\infty} = \{L_2 - L_1\}_{n=1}^{\infty}$$

is also null.

Now $\{L_2 - L_1\}_{n=1}^{\infty}$ is a constant null sequence. By exercise 6. page 74, $L_2 - L_1 = 0 \Rightarrow L_2 = L_1$. Thus the limit is unique.

Exercise

1. Show that the sequence whose general term is given below converges to the indicated number L.

   (a) $\dfrac{n+3}{n+4}, \quad L=1$

   (b) $\dfrac{n}{n^2+1}, \quad L=0$

   (c) $s_n = \begin{cases} 1 & \text{for } n \text{ odd} \\ \frac{n+1}{n} & \text{for } n \text{ even} \end{cases}, \quad L=1$

   (d) $\left(1+\dfrac{1}{n}\right)^3, \quad L=1$

   (e) $(-1)^n \dfrac{\pi}{n}, \quad L=0$

   (f) $\dfrac{4n^2-3}{3n^2+4}, \quad L=\dfrac{4}{3}$

2. Show that the sequence whose general term is given below is divergent.

   (a) $(-1)^n \dfrac{n}{n+1}$

   (b) $3n+5$

   (c) $\cos \dfrac{n\pi}{2}$

   (d) $n - \dfrac{1}{n}$

   (e) $s_n = \begin{cases} \frac{n+3}{n+4} & \text{for } n \text{ odd} \\ \frac{n+3}{2n+1} & \text{for } n \text{ even} \end{cases}$

Theorem 3.14. *If a sequence $\{s_n\}_{n=1}^{\infty}$ converges to L, the every subsequence converges to L also.*

Proof. Let $\{s_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{s_n\}_{n=1}^{\infty}$. So we have

$$n_1 < n_2 < n_3 < \ldots < n_k < \ldots$$

Let $\varepsilon > 0$ be given. As $s_n \to L$, there exists $N \in \mathbb{N}$ so that

$$|s_n - L| < \varepsilon, \qquad \forall n \geq N.$$

Now pick $K \in \mathbb{N}$ so that $n_K \geq N$. So if $k \geq K \Rightarrow n_k \geq n_K \geq N$. Thus

$$|s_{n_k} - L| < \varepsilon, \qquad \forall k \geq K.$$

This shows that $s_{n_k} \to L$.

Remark. This theorem is often used to show that a sequence diverges.

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence.

1. Suppose $\{s_{n_k}\}_{k=1}^{\infty}$ and $\{s_{n_l}\}_{l=1}^{\infty}$ are two subsequences of $\{s_n\}_{n=1}^{\infty}$, which have different limits,

$$s_{n_k} \to L_1 \qquad \text{and} \qquad s_{n_l} \to L_2 \quad (L_1 \neq L_2).$$

   Then sequence $\{s_n\}_{n=1}^{\infty}$ must be **divergent** by theorem 3.14. (Why?)

2. Suppose $\{s_{n_k}\}_{k=1}^{\infty}$ is a **divergent** subsequence then $\{s_n\}_{n=1}^{\infty}$ must be **divergent** also.

Example 3.11. These are two divergent sequences.

- $\{(-1)^n\}_{n=1}^{\infty}$

  Consider its two subsequences $\{(-1)^{2k}\}_{k=1}^{\infty}$ and $\{(-1)^{2l-1}\}_{l=1}^{\infty}$. They are constant sequences

$$\{(-1)^{2k}\}_{k=1}^{\infty} = \{1, 1, 1, \ldots\} \qquad \Rightarrow \qquad \lim_{k\to\infty} (-1)^{2k} = 1$$
$$\{(-1)^{2l-1}\}_{l=1}^{\infty} = \{-1, -1, -1, \ldots\} \qquad \Rightarrow \qquad \lim_{l\to\infty} (-1)^{2l-1} = -1$$

  $\{(-1)^n\}_{n=1}^{\infty}$ has subsequences which converge to different limits then $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

- $\{n \sin \dfrac{n\pi}{2}\}_{n=1}^{\infty}$

  Some first terms of the sequence are

$$1, 0, -3, 0, 5, 0, -7, 0, 9, 0, -11, \ldots$$

  Consider $s_1 = 1, s_5 = 5, s_9 = 9, \ldots, s_{n_k} = 4k-3, k = 1, 2, 3, \ldots$ This subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ is divergent. Then sequence $\{n \sin \dfrac{n\pi}{2}\}_{n=1}^{\infty}$ is also divergent.

Theorem 3.15. *Every convergent sequence is bounded.*

Proof. Suppose $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence with $s_n \to L$. By definition of convergent sequence, $\{s_n - L\}$ is null. Also by theorem 3.8 (page 73), every null sequence is bounded, i.e. there exists $M > 0$ so that $|s_n - L| < M$ for all n. Then

$$|s_n| = |(s_n - L) + L| \underset{\text{triangle inequality}}{\leq} |s_n - L| + |L| \leq M + |L| \leq \tilde{M}.$$

This shows that $\{s_n\}_{n=1}^{\infty}$ is bounded.

Remark. The converse of the statement is not true. ($\{s_n\}_{n=1}^{\infty}$ is bounded $\not\Rightarrow$ $\{s_n\}_{n=1}^{\infty}$ converges.) By example 3.11 (page 80), $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but it **does not** converge.

Example 3.12. Is sequence $\{n\}_{n=1}^{\infty}$ convergent?

If sequence $\{n\}_{n=1}^{\infty}$ is convergent then it must be bounded. However, $\{n\}_{n=1}^{\infty}$ is unbounded (check!) thus it **is not** convergent.

Exercise

1. Show that if a sequence $\{s_n\}_{n=1}^{\infty}$ has the property that $s_{2k} \to L$ and $s_{2k-1} \to L$, then $s_n \to L$.

2. Let $A = \{M | \text{where } |s_n| < M, \forall n\}$. Find the infimum of A ($\inf(A)$) where

   (a) $s_n = \dfrac{\sin n}{n}$

   (b) $s_n = \dfrac{4n^2 - 3}{3n^2 + 4}$

   (c) $s_n = \begin{cases} \frac{n+3}{n+4} & \text{for } n \text{ odd} \\ \frac{n+3}{2n+1} & \text{for } n \text{ even} \end{cases}$

   (d) $s_n = \begin{cases} 1 & \text{for } n \text{ odd} \\ \frac{n+1}{n} & \text{for } n \text{ even} \end{cases}$

   (e) $s_n = n - \dfrac{1}{n}$

Theorem 3.16. *If a sequence $\{s_n\}_{n=1}^{\infty}$ has a* ***positive limit****, then eventually the terms of the sequence become all* ***positive****.*

Proof. Suppose $s_n \to L > 0$. Then as $L > 0$, for $\varepsilon = L$, there exists $N \in \mathbb{N}$ so that

$$|s_n - L| < L, \qquad \forall n \geq N.$$

This means $-L < s_n - L < L$. Add L to the inequality,

$$0 < s_n < 2L, \qquad \Rightarrow \qquad s_n > 0 \quad \forall n \geq N.$$

Corollary 3.17. *If a sequence $\{s_n\}_{n=1}^{\infty}$ has a* ***negative limit****, then eventually the terms of the sequence become all* ***negative****.*

Proof. Exercise.

We can reformulate these two theorems:

Corollary 3.18. *Let $s = \{s_n\}_{n=1}^{\infty}$ be a sequence.*

1. *If there exists N so that $s_n \geq 0$ for all $n \geq N$, then s* ***cannot*** *have a* ***negative limit****.*

2. *If there exists N so that $s_n \leq 0$ for all $n \geq N$, then s* ***cannot*** *have a* ***positive limit****.*

Theorem 3.19. *Let $s = \{s_n\}_{n=1}^{\infty}$. If $\lim\limits_{n\to\infty} s_n = L$, then $\lim\limits_{n\to\infty} |s_n| = |L|$.*

Proof. Let $\varepsilon > 0$ be given. As $s_n \to L$, there exists $N \in \mathbb{N}$ so that

$$|s_n - L| < \varepsilon, \qquad \forall n \geq N.$$

By the second triangle inequality $||s_n| - |L|| \leq |s_n - L| < \varepsilon$, $\forall n \geq N$. Thus $|s_n| \to |L|$.

Remark. The converse of the statement is not true.

Example 3.13.

- $\lim_{n\to\infty} \frac{1-n}{n} = -1$. (check!)

  By theorem 3.19 (page 82), $\lim_{n\to\infty} \left|\frac{1-n}{n}\right| = \lim_{n\to\infty} \frac{n-1}{n} = 1$.

- Consider $s = \{(-1)^n\}_{n=1}^{\infty}$. We can see that $|s_n| = |(-1)^n| = 1, \forall n$ and $|s_n| \to 1$, while it has been shown in example 3.11 that s is a divergent sequence.

Theorem 3.20. *Suppose, $s_n \geq 0$ for all n, and $\lim_{n\to\infty} s_n = L$ then $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{L}$.*

Proof By the corollary 3.18 (page 82), L must be greater than or equal to zero.

- **Case I.** $L = 0$

  Let $\varepsilon > 0$ be given. Since $s_n \to 0$, there exists $N \in \mathbb{N}$ so that

$$
\begin{aligned}
|s_n - 0| = |s_n| = s_n &< \varepsilon^2, \qquad \forall n \geq N \\
\sqrt{s_n} &< \sqrt{\varepsilon^2} = \varepsilon \\
|\sqrt{s_n} - 0| &< \varepsilon
\end{aligned}
$$

- **Case II.** $L > 0$

  - **Discussion.**

$$
\begin{aligned}
|\sqrt{s_n} - \sqrt{L}| &< \varepsilon \\
\frac{|\sqrt{s_n} - \sqrt{L}||\sqrt{s_n} + \sqrt{L}|}{|\sqrt{s_n} + \sqrt{L}|} &< \varepsilon \\
|\sqrt{s_n} - \sqrt{L}||\sqrt{s_n} + \sqrt{L}| = |s_n - L| &< \varepsilon|\sqrt{s_n} + \sqrt{L}|
\end{aligned}
$$

  - **Real Proof.** Let $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ so that $|s_n - L| <$

$\varepsilon\sqrt{L}$ for all $n \geq N$.

$$\begin{aligned} |\sqrt{s_n} - \sqrt{L}| &= \frac{|\sqrt{s_n} - \sqrt{L}||\sqrt{s_n} + \sqrt{L}|}{|\sqrt{s_n} + \sqrt{L}|} \\ &= \frac{|s_n - L|}{|\sqrt{s_n} + \sqrt{L}|} \\ &< \frac{|s_n - L|}{\sqrt{L}} < \frac{\varepsilon\sqrt{L}}{\sqrt{L}} = \varepsilon \qquad \forall n \geq N \end{aligned}$$

The two above cases show that $\sqrt{s_n} \to \sqrt{L}$.

The following theorem is important in the discussion of series.

Theorem 3.21. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. Since $n \geq 1$, we get $\sqrt[n]{n} \geq 1$ (check!). So we can write

$$\begin{aligned} \sqrt[n]{n} &= 1 + p_n, \qquad p_n \geq 0 \\ n = (\sqrt[n]{n})^n &= (1 + p_n)^n \\ &\underset{\text{By binomial theorem}}{=} 1 + \binom{n}{1}p_n + \binom{n}{2}(p_n)^2 + \cdots + \binom{n}{n}(p_n)^n \\ &= 1 + np_n + \frac{n(n-1)}{2}(p_n)^2 + \cdots + (p_n)^n \end{aligned}$$

Since each term of the right hand side of the equation is greater than or equal to zero then

$$n \geq \frac{n(n-1)}{2}(p_n)^2 \geq 0.$$

So the square root of p_n becomes

$$0 \leq p_n \leq \sqrt{\frac{2}{n-1}}$$

Since $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is a null sequence thus $\left\{\frac{1}{n-1}\right\}_{n=2}^{\infty}$ is null also. By theorem 3.6 (page 73.), $\left\{\sqrt{\frac{2}{n-1}}\right\}_{n=2}^{\infty}$ is a null sequence.

Since $0 \leq p_n = \sqrt[n]{n} - 1 \leq \sqrt{\frac{2}{n-1}}$, by squeeze theorem (page 74), sequence $\{p_n = \sqrt[n]{n} - 1\}_{n=1}^{\infty}$ is null. Thus

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Remark. In this proof, we have used the fact that if we change or ignore the first terms in a sequence, convergence and limits are unaffected.

Theorem 3.22 (Squeeze theorem of limits)**.** *Let* $\{r_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *be two sequences, and* $\{s_n\}_{n=1}^{\infty}$ *is a sequence with*

$$r_n \leq s_n \leq t_n \qquad \forall n.$$

If $\lim_{n\to\infty} r_n = \lim_{n\to\infty} t_n = L$ *then* $\lim_{n\to\infty} s_n = L$ *also.*

Proof. As $r_n \to L$ and $t_n \to L$, we know that

$$\{r_n - L\}_{n=1}^{\infty} \qquad \text{and} \qquad \{t_n - L\}_{n=1}^{\infty}$$

are both **null** sequence. Since $r_n \leq s_n \leq t_n, \forall n \Rightarrow \underbrace{r_n - L}_{\text{null}} \leq s_n - L \leq \underbrace{t_n - L}_{\text{null}}, \forall n.$ By squeeze theorem (page 74), $s_n - L$ is a null sequence also. Thus

$$s_n \to L.$$

Exercise

1. Prove corollary 3.17 (page 82).

2. Show that if a sequence $\{s_n\}_{n=1}^{\infty}$ has the property that $s_{2k} \to L$ and $s_{2k-1} \to L$, then $s_n \to L$.

3. Show that a sequence $\{s_n\}_{n=1}^{\infty}$ converges to a number L if and only if, for every $m \in \mathbb{N}$, the tail $\{s_n\}_{n=m}^{\infty}$ converges to L.

4. The limit of a sequence $\{s_n\}_{n=1}^{\infty}$ does not change if we change finitely many terms s_n of the sequence.

5. Show that if $a > 0$, $\lim_{n\to\infty} \sqrt[n]{a} = 1$. (**Hint.** Apply squeeze theorem to prove this statement.)

3.5 Cauchy Sequences

Definition 3.10 (Cauchy Sequence)**.** A sequence $\{s_n\}_{n=1}^{\infty}$ is called **Cauchy** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that

$$|s_n - s_m| < \varepsilon, \qquad \text{for all } n, m \geq N.$$

Example 3.14. Sequence $\left\{\dfrac{n+2}{n}\right\}_{n=1}^{\infty}$ is Cauchy.

Let $\varepsilon > 0$ be given. We want :

$$\begin{aligned} |s_n - s_m| &< \varepsilon \\ \left|\frac{n+2}{n} - \frac{m+2}{m}\right| &< \varepsilon \\ \left|\left(1+\frac{2}{n}\right) - \left(1+\frac{2}{m}\right)\right| &< \varepsilon \\ \left|\frac{2}{n} - \frac{2}{m}\right| &< \varepsilon \end{aligned}$$

Since $\left|\dfrac{2}{n} - \dfrac{2}{m}\right| = \left|\dfrac{2}{n} + \left(-\dfrac{2}{m}\right)\right| \leq \left|\dfrac{2}{n}\right| + \left|-\dfrac{2}{m}\right| = \left|\dfrac{2}{n}\right| + \left|\dfrac{2}{m}\right| = \dfrac{2}{n} + \dfrac{2}{m}$

The inequality $\left|\dfrac{2}{n} - \dfrac{2}{m}\right| < \varepsilon$ is true if $\dfrac{2}{n} < \dfrac{\varepsilon}{2}$ and $\dfrac{2}{m} < \dfrac{\varepsilon}{2}$, i.e

$$n > \frac{4}{\varepsilon} \qquad \text{and} \qquad m > \frac{4}{\varepsilon}.$$

Choose $N \in \mathbb{N}$ so that $N > \dfrac{4}{\varepsilon}\left(\Rightarrow \dfrac{4}{N} < \varepsilon\right)$. Then if $m, n \geq N$, we have

$$|s_n - s_m| = \left|\frac{2}{n} - \frac{2}{m}\right| \leq \frac{2}{n} + \frac{2}{m} \leq \frac{2}{N} + \frac{2}{N} = \frac{4}{N} < \frac{4}{\frac{4}{\varepsilon}} = \varepsilon.$$

This shows that $\left\{\dfrac{n+2}{n}\right\}_{n=1}^{\infty}$ is Cauchy.

Lemma 3.23. *Every convergent sequence is Cauchy.*

Proof. Suppose $\{s_n\}_{n=1}^{\infty}$ is convergent, say $s_n \to L$. Let $\varepsilon > 0$ be given. Since $s_n \to L$, there exists $N \in \mathbb{N}$ so that

$$|s_n - L| < \frac{\varepsilon}{2}, \qquad \forall n \geq N.$$

$$\text{Also} \qquad |s_m - L| < \frac{\varepsilon}{2}, \qquad \forall m \geq N.$$

Then

$$\begin{aligned} |s_n - s_m| = |(s_n - L) - (s_m - L)| &\leq |(s_n - L)| + |-(s_m - L)| \\ &= |s_n - L| + |s_m - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Lemma 3.24. *Every Cauchy sequence is bounded.*

Proof. As $\{s_n\}_{n=1}^{\infty}$ is Cauchy, for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ so that

$$|s_n - s_m| < \varepsilon = 1, \qquad \text{for } m, n \geq N.$$

Choose $m = N$, so $|s_n - s_N| < 1$ for all $n \geq N$.

$$|s_n| = |s_n - s_N + s_N| \leq |s_n - s_N| + |s_N| < 1 + |s_N|, \qquad \text{for } n \geq N.$$

Set $M = \max\{|s_1|, |s_2|, \ldots, |s_{N-1}|, |s_N| + 1\}$. Then for all $n \in \mathbb{N}, |s_n| \leq M$. Thus $\{s_n\}_{n=1}^{\infty}$ is bounded.

Lemma 3.25. *If $\{s_n\}_{n=1}^{\infty}$ is Cauchy and $\{s_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence, say $s_{n_k} \to L$, then $\{s_n\}_{n=1}^{\infty}$ itself converges to L.*

Proof. Let $\varepsilon > 0$ be given. As $\{s_n\}$ is Cauchy, there exits $N \in \mathbb{N}$ so that $|s_n - s_m| < \frac{\varepsilon}{2}, \forall m, n \geq N$. As $s_{n_k} \to L$, there exits $K \in \mathbb{N}$ so that $|s_{n_k} - L| < \frac{\varepsilon}{2}, \forall k \geq K$.

Fix an index K_1 so that $K_1 \geq K$ and $n_{K_1} \geq N$. So if $n \geq N$,

$$\begin{aligned} |s_n - L| &= |s_n - s_{n_{K_1}} + s_{n_{K_1}} - L| \\ &\leq \underbrace{|s_n - s_{n_{K_1}}|}_{\text{Cauchy seq.}} + \underbrace{|s_{n_{K_1}} - L|}_{\text{convergent seq.}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $s_n \to L$.

Theorem 3.26 (Cauchy Criterion). *A sequence of **real numbers** converges if and only if it is Cauchy.*

Proof. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence.

($\Rightarrow$) Suppose $\{s_n\}_{n=1}^{\infty}$ converges. By lemma 3.23 (page 86.), it is Cauchy.

($\Leftarrow$) Suppose $\{s_n\}_{n=1}^{\infty}$ is Cauchy, by lemma 3.24 (page 87.), it is bounded. By the Bolzano-Weierstrass theorem[1] $\{s_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{s_{n_k}\}_{k=1}^{\infty}$.

By lemma 3.25 (page 87.), the sequence $\{s_n\}_{n=1}^{\infty}$ itself converges.

Remarks.

1. The notion of a Cauchy sequence is useful because it allows us to discuss convergent sequences without knowing their limit.

2. Theorem 3.26 **does not hold** if we work in the set $\mathbb{Q}$ of rational numbers, while lemma 3.23,3.24 and 3.25 hold (check!) because the Bolzano-Weierstrass theorem does not hold in $\mathbb{Q}$.

Exercise

Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be Cauchy sequences. Determine whether the following sequences are Cauchy or not.

1. $\{s_n + t_n\}$.
2. $\{s_n - t_n\}$.
3. $\{s_n t_n\}$.
4. $\left\{\dfrac{s_n}{t_n}\right\}$, where $t_n \neq 0, \forall n$.

[1] **Bolzano-Weierstrass theorem.** *Every bounded sequence of real numbers has a convergent subsequence.*

The proof of this theorem will be discussed later (see the proof in page 105).

Theorem 3.27. *Let* $\{s_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *be convergent sequences, say*

$$\lim_{n\to\infty} s_n = L \qquad \text{and} \qquad \lim_{n\to\infty} t_n = M.$$

then $\{s_n + t_n\}_{n=1}^{\infty}, \{s_n - t_n\}_{n=1}^{\infty}$ *and* $\{s_n t_n\}_{n=1}^{\infty}$ *are convergent also and*

1. $\lim_{n\to\infty} (s_n + t_n) = L + M$.

2. $\lim_{n\to\infty} (s_n - t_n) = L - M$.

3. $\lim_{n\to\infty} (s_n t_n) = LM$.

Proof.

1. As $s_n \to L$ and $t_n \to M$, the sequences $\{s_n - L\}$ and $\{t_n - M\}$ are null. By theorem 3.4 (page 72.), the sum of these two null sequences are also null, i.e.
$$\left\{ \underbrace{(s_n - L)}_{\text{null}} + \underbrace{(t_n - M)}_{\text{null}} \right\} = \{(s_n + t_n) - (L + M)\}.$$
By the definition, it shows that $s_n + t_n \to L + M$.

2. Exercise.

3. Note that
$$\begin{aligned} s_n t_n - LM &= s_n t_n - t_n L + t_n L - LM \\ &= t_n(s_n - L) + L(t_n - M) \end{aligned}$$
Since t_n is convergent sequence then it is bounded (by theorem 3.15, page 81). As $\{s_n - L\}$ is null, $\{t_n(s_n - L)\}$ is null also (by theorem 3.6, page 73). As L is a constant, by corollary 3.7 (page 73.), $\{L(t_n - M)\}$ is a null sequence.

   Since $\{s_n t_n - LM\}$ can be expressed as a sum of null sequences, then it is a null sequence also. This means $s_n t_n \to LM$.

Remark. This theorem may be expressed in the other way as :

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences :

1. $\lim_{n\to\infty}(s_n \pm t_n) = \lim_{n\to\infty} s_n \pm \lim_{n\to\infty} t_n$.

2. $\lim_{n\to\infty} s_n t_n = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$.

We want a similar theorem for sequences of quotients $\dfrac{s_n}{t_n}$. However, we have to be aware for the case that $t_n = 0$ for some n or $t_n \to 0$.

Lemma 3.28. *If $t_n \neq 0$ for all $n \in \mathbb{N}$ and $t_n \to M \neq 0$, then the sequence $\left\{\dfrac{1}{t_n}\right\}_{n=1}^{\infty}$ is bounded.*

Proof. Since $t_n \to M$, we have $|t_n| \to |M|$ (theorem 3.15, page 81.). By the assumption $M \neq 0$, set $\varepsilon = \dfrac{|M|}{2}$. There exists $N \in \mathbb{N}$ so that $||t_n| - |M|| < \dfrac{|M|}{2}, \forall n \geq N$ so

$$\begin{aligned} -\frac{|M|}{2} < \quad |t_n| - |M| \quad &< \frac{|M|}{2} \\ \frac{|M|}{2} < \qquad |t_n| \qquad &< 3\frac{|M|}{2}, \qquad \forall n \geq N. \end{aligned}$$

Take reciprocals

$$\frac{2}{|M|} > \frac{1}{|t_n|} > 3\frac{2}{|M|}$$

Thus $\left|\dfrac{1}{t_n}\right| < \dfrac{2}{|M|}$ for all $n \geq N$.

Set $B = \max\left\{\left|\dfrac{1}{t_1}\right|, \left|\dfrac{1}{t_2}\right|, \ldots, \left|\dfrac{1}{t_{N-1}}\right|, \dfrac{2}{|M|}\right\}$ then $\left|\dfrac{1}{t_n}\right| < B+1$ for all $n \in \mathbb{N}$.

This shows that $\left\{\dfrac{1}{t_n}\right\}_{n=1}^{\infty}$ is bounded.

Theorem 3.29. *Suppose $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} t_n = M$. If $t_n \neq 0, \forall n$ and $M \neq 0$, then $\left\{\dfrac{s_n}{t_n}\right\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} \dfrac{s_n}{t_n} = \dfrac{L}{M}$*

Proof. Note that

$$\begin{aligned}
\frac{s_n}{t_n} - \frac{L}{M} &= \frac{Ms_n - t_nL}{t_nM} \\
&= \frac{1}{t_n}\left(\frac{Ms_n - ML + ML - t_nL}{M}\right) \\
&= \frac{1}{t_n}\left(\frac{M(s_n - L) + (M - t_n)L}{M}\right) \\
&= \frac{1}{t_n}\left((s_n - L) + \frac{L}{M}(M - t_n)\right) \\
&= \frac{1}{t_n}\left((s_n - L) - \frac{L}{M}(t_n - M)\right)
\end{aligned}$$

As $s_n \to L, \{s_n - L\}$ is null. As $t_n \to M, \{t_n - M\}$ and $\dfrac{L}{M}(t_n - M)$ are null. By the theorem on null sequence (page 72.), $\left\{(s_n - L) - \frac{L}{M}(t_n - M)\right\}_{n=1}^{\infty}$ is null. By lemma 3.28 (page 90.) $\left\{\dfrac{1}{t_n}\right\}_{n=1}^{\infty}$ is bounded so $\left\{\dfrac{1}{t_n}\left((s_n - L) - \dfrac{L}{M}(t_n - M)\right)\right\}_{n=1}^{\infty}$ is also null. This means $\left\{\dfrac{s_n}{t_n} - \dfrac{L}{M}\right\}_{n=1}^{\infty}$ is null, so

$$\frac{s_n}{t_n} \to \frac{L}{M}.$$

Exercise

1. Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be convergent sequences, say

$$\lim_{n\to\infty} s_n = L \qquad \text{and} \qquad \lim_{n\to\infty} t_n = M.$$

   Show that $\{s_n - t_n\}_{n=1}^{\infty}$ is convergent also and $\underset{n\to\infty}{lim}(s_n - t_n) = L - M$.

2. Determine whether the given sequence converges; if it does find the limit.

   (a) $\sqrt{\dfrac{n^2+4}{9n^2+3}}$

   (b) $\dfrac{1+3^n}{1+2^n}$

   (c) $\dfrac{1+2^n}{3^n+2^n}$

3.6 Divergence to Infinity

Unbounded sequences are necessarily divergent (Theorem 3.15, page 81.) In this section we investigate two special types of unbounded sequences.

3.6.1 Positive Infinity

Definition 3.30. A sequence $\{s_n\}_{n=1}^{\infty}$ **diverges to infinity** if for every $B > 0$ there exists $N \in \mathbb{N}$ so that $s_n > B$ for all $n \geq N$.

In this case we write

$$\lim_{n\to\infty} s_n = \infty \qquad \text{or} \qquad s_n \to \infty.$$

Roughly speaking, this means we can make the eventual terms of the sequence as large as we wish ($s_n > B$) by choosing n sufficiently large ($n \geq N$).

Example 3.15. Consider the following sequences

- Sequence $\{s_n\}_{n=1}^{\infty} = \left\{n^3 - 2n^2\right\}_{n=1}^{\infty}$ diverges to infinity.

  Let $B > 0$ be given. We want to find n so that $n^3 - 2n^2 > B \Rightarrow n^2(n - 2) > B$. This is true if $n^2 > B$ and $n - 2 \geq 1$. So choose $N \in \mathbb{N}$ with $N > \max\{\sqrt{B}, 3\}$. If $n \geq N$ then $n^3 - 2n^2 = n^2(n-2) \geq N^2(N-2) > \sqrt{(B)}^2 \cdot 1 = B$. Hence $n^3 - 2n^2 > B, \forall n \geq N$. This shows that $s_n \to \infty$.

- Sequence $\{s_n\}_{n=1}^{\infty} = \{n\left(1 + (-1)^n\right)\}_{n=1}^{\infty}$.

  The terms of sequence are

  $$0, 4, 0, 8, 0, 12, 0, \ldots$$

  This sequence is unbounded. However, it does not diverge to infinity, as infinitely many terms are zero.

Theorem 3.31. *If $s_n \to \infty$ and $t_n \to \infty$ then*

1. $s_n + t_n \to \infty$

2. $s_n \cdot t_n \to \infty$

Proof. Exercise.

Theorem 3.32. *If $s_n \to \infty$ and $\{t_n\}_{n=1}^{\infty}$ is bounded then $s_n + t_n \to \infty$.*

Proof. Exercise.

Since all convergent sequences are bounded then we get

Corollary 3.33. *If $s_n \to \infty$ and $\{t_n\}_{n=1}^{\infty}$ is convergent then $s_n + t_n \to \infty$.*

Proof. Exercise.

3.6.2 Negative Infinity

In a similar way, we define divergent to $-\infty$.

Definition 3.11. A sequence $\{s_n\}_{n=1}^{\infty}$ **diverges to negative infinity** if for every $B < 0$ there exists $N \in \mathbb{N}$ so that $s_n < B$ for all $n \geq N$.

In this case we write

$$\lim_{n\to\infty} s_n = -\infty \qquad \text{or} \qquad s_n \to -\infty.$$

Example 3.16. Consider the following sequences

- Sequence $\{s_n\}_{n=1}^{\infty} = \left\{10n - n^2\right\}_{n=1}^{\infty}$ diverges to negative infinity.

  Let $B < 0$ be given. We want

$$\begin{aligned} 10n - n^2 &< B \\ n(10 - n) &< B \\ n(n - 10) &> -B \end{aligned}$$

The inequality is true if $n > -B$ and $n \geq 11$. Choose $N \in \mathbb{N}$ so that $N > -B$ and $N \geq 11$ so if $n \geq N$ then

$$n(n-10) \geq N(N-10) \geq N > -B.$$

Multiply the inequality by -1

$$-n(n-10) = n(10-n) < -(-B) = B, \qquad \forall n \geq N.$$

This shows that $10n - n^2 \to -\infty$.

- Sequence $\{s_n\}_{n=1}^{\infty} = \{(-1)^n n\}_{n=1}^{\infty}$.

  The terms of sequence are

$$-1, 2, -3, 4, -5, 6, -7, 8, \ldots$$

  If we consider some subsequences $\{s_{2k-1}\}_{k=1}^{\infty}$ and $\{s_{2k}\}_{k=1}^{\infty}$, we found that

$$\text{terms of sequence } \{s_{2k-1}\} \text{ are} \qquad -1, -3, -5, -7, \ldots \to -\infty$$
$$\text{terms of sequence } \{s_{2k}\} \text{ are} \qquad 2, 4, 6, 8, \ldots \to \infty$$

  Subsequences $\{s_{2k-1}\}$ and $\{s_{2k}\}$ diverge to **negative infinity** and to **positive infinity**, respectively. However, $\{s_n\}_{n=1}^{\infty}$ does **not** diverge neither to **negative infinity** nor to **positive infinity**.

Theorem 3.34. *$s_n \to -\infty$ if and only if $-s_n \to \infty$.*

Proof. Exercise.

Theorem 3.35. *If $s_n \to -\infty$ and $t_n \to -\infty$ then*

1. $s_n + t_n \to -\infty$

2. $s_n \cdot t_n \to \infty$

Proof. Exercise.

Theorem 3.36. *If* $s_n \to -\infty$ *and* $\{t_n\}_{n=1}^{\infty}$ *is bounded then* $s_n + t_n \to -\infty$*.*

Proof. Exercise.

Corollary 3.37. *If* $s_n \to -\infty$ *and* $\{t_n\}_{n=1}^{\infty}$ *is convergent then* $s_n + t_n \to -\infty$*.*

Proof. Exercise.

Theorem 3.38. *Suppose* $s_n \to \infty$ *and* $t_n \to L$

1. *If* $L > 0$*, then* $s_n t_n \to \infty$*.*
2. *If* $L < 0$*, then* $s_n t_n \to -\infty$*.*
3. *If* $L = 0$*, then we cannot say anything about the convergence of* $s_n t_n$*.*

Proof.

1. First, make sure that t_n is sufficiently different from zero. Choose $\varepsilon = \dfrac{L}{2}$.

   As $t_n \to L$, there exists $N_1 \in \mathbb{N}$ so that

$$\begin{aligned} |t_n - L| &< \frac{L}{2}, \qquad \forall n \geq N_1 \\ \text{i.e.} \quad -\frac{L}{2} < t_n - L &< \frac{L}{2} \\ \frac{L}{2} < t_n &< 3\frac{L}{2} \end{aligned}$$

   This shows that $t_n > \dfrac{L}{2}$, for all $n \geq N$.

   Next, claim $s_n t_n \to \infty$.

   Let $B > 0$ be given. As $s_n \to \infty, \exists N_2 \in \mathbb{N}$ so that $s_n > 2\dfrac{B}{L}$ for all $n \geq N_2$.

   Set $N = \max\{N_1, N_2\}$, if $n \geq N$ then we get

$$s_n \cdot t_n > 2\frac{B}{L} \cdot \frac{L}{2} = B.$$

   As $B > 0$ is arbitrary, this proves the claim that $s_n t_n \to \infty$.

2. Similarly.

3. Exercise.

Theorem 3.39. *Let* $\{s_n\}_{n=1}^{\infty}$ *be a sequence with* $s_n \neq 0$ *for all* n*. Then* $|s_n| \to \infty$ *if and only if* $\frac{1}{s_n} \to 0$.

Proof.

$\Rightarrow$ Assume $|s_n| \to \infty$. Let $\varepsilon > 0$ be given. As $|s_n| \to \infty$, there exists $N \in \mathbb{N}$ so that $|s_n| > \frac{1}{\varepsilon}$ for all $n \geq N$.

The reciprocals

$$\frac{1}{|s_n|} < \varepsilon \Rightarrow \left|\frac{1}{s_n}\right| < \varepsilon, \qquad \forall n \geq N.$$

As $\varepsilon > 0$ is arbitrary we see that $\frac{1}{s_n} \to 0$.

$\Leftarrow$ Suppose $\frac{1}{s_n} \to 0$. Let $B > 0$ be arbitrary. As $\frac{1}{s_n} \to 0$, there exists $N \in \mathbb{N}$ so that

$$\left|\frac{1}{s_n}\right| < \frac{1}{B} \qquad \Rightarrow \qquad |s_n| > B, \forall n \geq N.$$

As B is arbitrary, this shows that $|s_n| \to \infty$.

Theorem 3.40. *Consider the geometric sequence* $\{r^n\}_{n=1}^{\infty}$ *where* r *is fixed.*

1. If $|r| < 1$*, then* $r^n \to 0$.

2. If $r = 1$*, then* $r^n \to 1$.

3. If $r > 1$*, then* $r^n \to \infty$.

4. If $r \leq -1$*, then* $\{r^n\}_{n=1}^{\infty}$ *diverges.*

Proof.

1. This is Theorem 3.12 (Page 75.).

2. Obvious.

3. Suppose $r > 1$, then $0 < \frac{1}{r} < 1$. By part 1, $\left(\frac{1}{r}\right)^n \to 0$. By theorem 3.39 (page 96.) $|r^n| = |r|^n = r^n \to \infty$.

4. Suppose $r = -1$ then $r^n = (-1)^n$. Clearly $(-1)^n$ diverges. Suppose $r < -1$ then $-r > 1$. By part 3, $|r^n| = |r|^n \to \infty$. Thus sequence $\{|r^n|\}$ is unbounded. It follows that $\{r^n\}$ is unbounded. So r^n must diverges. However, r^n has alternating signs so r^n does not diverge neither to ∞ nor to $-\infty$.

Theorem 3.41. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence which diverges to ∞ (or $-\infty$). Then **every** subsequence $\{s_n\}_{n_k}$ diverges to ∞ (or $-\infty$) also.*

Proof. Exercise.

Exercise

1. Prove theorem 3.31.

2. Prove theorem 3.32.

3. Prove corollary 3.33.

4. Prove theorem 3.34

5. Prove theorem 3.35

6. Prove theorem 3.36

7. Prove corollary 3.37

8. Prove theorem 3.38 part 3.

9. Prove theorem 3.41.

The following theorem is extremely important and follows directly from the completeness property of real numbers $\mathbb{R}$

Theorem 3.42.

1. *If* $\{s_n\}_{n=1}^{\infty}$ *is* ***increasing*** *and* ***bounded above****, then it is convergent, and* $\lim_{n\to\infty} s_n = \sup\{s_n | n \in \mathbb{N}\}$.
2. *If* $\{s_n\}_{n=1}^{\infty}$ *is* ***decreasing*** *and* ***bounded below****, then it is convergent, and* $\lim_{n\to\infty} s_n = \inf\{s_n | n \in \mathbb{N}\}$.
3. *A bounded monotone sequence is convergent.*

Proof.

1. Let $\{s_n\}_{n=1}^{\infty}$ be increasing and bounded above. Set $L = \sup\{s_n | n \in \mathbb{N}\}$.

   Claim: $s_n \to L$.

   Let $\varepsilon > 0$ be given. By theorem 2.3 (page 38.) there exists an element $s_N \in \{s_n | n \in \mathbb{N}\}$ so that

   $$L - \varepsilon < s_N \leq L.$$

   However $\{s_n\}$ is increasing

   $$\begin{aligned} L - \varepsilon < s_N \leq s_{N+1} \leq s_{N+2} \leq \ldots \leq \quad & s_n \quad \leq \ldots \leq L, \quad \forall n \geq N \\ L - \varepsilon < \quad & s_n \quad \leq L < L + \varepsilon, \qquad \forall n \geq N \\ -\varepsilon < \quad & s_n - L \quad < \varepsilon \\ |s_n - L| \quad & < \quad \varepsilon, \qquad \forall n \geq N \end{aligned}$$

   This shows that $s_n \to L$.

2. Similar.

3. Follows from part 1 and 2.

Corollary 3.43. *Sequence* $\{s_n\}_{n=1}^{\infty} = \left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ *is convergent.*

Proof. Claim : The sequence $\{s_n\}_{n=1}^{\infty}$ is **increasing**.

By the binomial theorem

$$\begin{aligned}
s_n = \left(1+\frac{1}{n}\right)^n &= \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} \left(\frac{1}{n}\right)^i = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{n^i} \\
&= \sum_{i=0}^{n} \underbrace{\frac{n(n-1)\cdots(n-i+1)}{i!}}_{i \text{ terms}} \frac{1}{n^i} \\
&= \sum_{i=0}^{n} \frac{1}{i!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-i+1}{n} \\
&= \sum_{i=0}^{n} \frac{1}{i!}\, 1 \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{i-1}{n}\right) \\
&= \sum_{i=0}^{n} \frac{1}{i!} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{i-1}{n}\right) \\
\text{also} \qquad s_{n+1} &= \sum_{i=0}^{n+1} \frac{1}{i!} \left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{i-1}{n+1}\right)
\end{aligned}$$

Consider $s_{n+1} - s_n$, which is

$$\begin{aligned}
&\sum_{i=0}^{n} \frac{1}{i!} \left[\underbrace{\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{i-1}{n+1}\right)}_{(a)} - \underbrace{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{i-1}{n}\right)}_{(b)} \right] \\
&+\frac{1}{(n+1)!} \left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right) \qquad (3.7)
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{n+1} < \frac{1}{n} \quad &\Rightarrow \quad 1-\frac{1}{n+1} > 1-\frac{1}{n} \\
\frac{2}{n+1} < \frac{2}{n} \quad &\Rightarrow \quad 1-\frac{2}{n+1} > 1-\frac{2}{n} \\
&\vdots \\
\frac{i}{n+1} < \frac{i}{n} \quad &\Rightarrow \quad 1-\frac{i}{n+1} > 1-\frac{i}{n} \\
&\Rightarrow \quad (a) > (b) \Rightarrow (a)-(b) > 0
\end{aligned}$$

thus every term of summation 3.7 is positive. This implies that $s_{n+1} - s_n > 0 \Rightarrow$ $\{s_n\}_{n=1}^{\infty}$ is increasing.

Claim : The sequence $\{s_n\}_{n=1}^{\infty}$ is **bounded above**.

Since $0 < \left(1 - \frac{i}{n}\right) < 1, i = 1, \ldots, n-1$ then

$$s_n = \sum_{i=0}^{n} \frac{1}{i!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{i-1}{n}\right) < \sum_{i=0}^{n} \frac{1}{i!}$$

Consider

$$\begin{aligned}
i! &= i(i-1)(i-2)\cdots 3 \cdot 2 \cdot 1 \\
&\geq 2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 \cdot 1 = 2^{i-1} \\
\Rightarrow \quad \frac{1}{2^{i-1}} &\geq \frac{1}{i!} \\
\Rightarrow \quad s_n &\leq \sum_{i=0}^{n} \frac{1}{i!} = \frac{1}{0!} + \sum_{i=1}^{n} \frac{1}{i!} \\
&\leq 1 + \sum_{i=1}^{n} \frac{1}{2^{i-1}} = 1 + \underbrace{\sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i-1}}_{\text{geometry series}} < 1 + 2 = 3
\end{aligned}$$

So $s_n \leq 3 \Rightarrow \{s_n\}_{n=1}^{\infty}$ is bounded above.

As $\{s_n\}_{n=1}^{\infty}$ is increasing and bounded above then $\{s_n\}_{n=1}^{\infty}$ is convergent.

The limit of this sequence is called the number e.

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n,$$

where $e \approx 2.71829\ldots$.

Theorem 3.44.

1. *If $\{s_n\}_{n=1}^{\infty}$ is **increasing** then $\{s_n\}_{n=1}^{\infty}$ either converges or diverges to ∞.*

2. *If $\{s_n\}_{n=1}^{\infty}$ is **decreasing** then $\{s_n\}_{n=1}^{\infty}$ either converges or diverges to $-\infty$.*

3. *A monotone sequence either converges or diverges to ∞ or $-\infty$.*

Proof.

1. Let $\{s_n\}_{n=1}^{\infty}$ be increasing. If $\{s_n\}$ is bounded above then by theorem 3.42, $\{s_n\}$ converges. If $\{s_n\}$ is not bounded above, then for any given $B > 0$ there exists a term s_N so that $s_N > B$. However the sequence is increasing, so $s_N \leq s_{N+1} \leq s_{N+2} \leq \ldots$. Thus

$$s_n > B \qquad \forall n \geq N.$$

   As B is an arbitrary, this show that $s_n \to \infty$.

2. Similar.

3. Follows from part 1 and 2.

Theorem 3.45. *Let S be a non-empty set of real numbers which is bounded above. Let $L = \sup S$. Then there exists a sequence $\{s_n\}_{n=1}^{\infty}$ of elements of S such that $s_n \to L$.*

Proof. By theorem 2.3 (page 38.), for any $\varepsilon > 0$, there exists $s \in S$ so that $L - \varepsilon < s \leq L$. Thus for every $n \in \mathbb{N}$, there exists an element $s_n \in S$ so that $L - \dfrac{1}{n} < s_n \leq L$. Consider the sequence $\{s_n\}_{n=1}^{\infty}$. We know that $\lim\limits_{n\to\infty} \left(L - \frac{1}{n}\right) = L$, by the squeeze theorem $\lim\limits_{n\to\infty} s_n = L$ also.

By an idea of the above theorem combining with theorem 3.42, *"if $\{s_n\}_{n=1}^{\infty}$ is* ***increasing*** *and* ***bounded above****, then it is convergent"*, it gives us the following theorem.

Theorem 3.46. *Let S be a non-empty set of real numbers which is bounded above. Suppose $L = \sup S$ which is $L \notin S$. Then there exists a* ***strictly increasing*** *sequence $\{s_n\}_{n=1}^{\infty}$ of elements of S such that $s_n \to L$.*

Proof. Pick s_n as follows :

1. Choose s_1 to be any element in S so that $L-1 < s_1 < L$.

2. Suppose $s_1, s_2, \ldots, s_n$ have already been chosen so that

$$s_1 < s_2 < s_3 < \ldots < s_n \quad \text{and} \quad L - \frac{1}{k} < s_k < L, \text{ for } k = 1, 2, \ldots, n.$$

   To choose s_{n+1}, by theorem 2.3 (page 38.), there exists $\dot{s} \in S$ so that

$$L - \frac{1}{n+1} < \dot{s} < L.$$

   Because $s_n < L$,also by the same theorem, there exists $\ddot{s} \in S$ so that

$$s_n < \ddot{s} < L.$$

   Choose $s_{n+1} = \max(\dot{s}, \ddot{s})$ then $s_n < s_{n+1}$ which is

$$L - \frac{1}{n+1} < s_{n+1} < L.$$

By mathematical induction, for every $n \in \mathbb{N}$, we can choose $s_n \in S$ so that

1. $\{s_n\}_{n=1}^{\infty}$ is increasing.

2. $L - \dfrac{1}{n} < s_n < L$

By 2 and squeeze theorem, $\lim\limits_{n\to\infty} s_n = L$.

Exercise

1. Show that $\lim\limits_{n\to\infty} \left(1 + \dfrac{2}{n}\right)^n = e^2$.

2. Show that $\lim\limits_{n\to\infty} \left(1 - \dfrac{1}{n}\right)^n = \dfrac{1}{e}$.

3. Show that $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{2n}\right)^n = \sqrt{e}$.

4. Let $\{s_n\}_{n=1}^{\infty}$ be defined recursively by

(a) $s_1 = 1$

(b) $s_{n+1} = \sqrt{1 + s_n},\ \forall n \in \mathbb{N}$.

Show that $\{s_n\}_{n=1}^{\infty}$ is convergent and find $\lim_{n\to\infty} s_n$.

3.6.3 The Bolzano-Weierstrass Theorem

Lemma 3.47. *Every sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ has a monotone subsequence.*

Proof. We can list the elements of the sequence by

$$s : s_1, s_2, s_3, \ldots$$

Consider the tails of the sequence by write down subsequences obtained by cutting the first, the second, the third element, and so on :

$$\begin{aligned} s^{(1)} &: s_1, s_2, s_3, \ldots \\ s^{(2)} &: s_2, s_3, s_4, \ldots \\ s^{(3)} &: s_3, s_4, s_5, \ldots \\ &\vdots \\ s^{(n)} &: s_n, s_{n+1}, s_{n+2}, \ldots \\ &\vdots \end{aligned}$$

We distinguish two cases : Either **Case I.** every tail $s^{(n)}$ has a largest element, or **Case II.** at least one of these tails has no largest element.

- **Case I.** Every tail $s^{(n)}$ has a largest element.

  In this case, $\max(s^{(k)} = \max\{s_n\}_{n=k}^{\infty}$ exists for **all** k. Construct a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ as follows :

1. Let n_1 be the smallest index so that $s_{n_1} = \max\{s_n\}_{n=1}^{\infty}$. (There may exist many values of $\bar{n}$ so that $s_{\bar{n}} = \max\{s_n\}_{n=1}^{\infty}$, we let n_1 be the **smallest** of such $\bar{n}$.)

2. To find s_{n_2}, we look at the tail after the element n_1, i.e. s^{n_1+1} Let n_2 be the **smallest** index so that $s_{n_2} = \max\{s_n\}_{n=n_1+1}^{\infty}$. Then we have $n_1 < n_2$ and $s_{n_1} \geq s_{n_2}$.

3. Next let n_3 be the **smallest** index so that $s_{n_3} = \max\{s_n\}_{n=n_2+1}^{\infty}$.

4. In general, suppose we have picked indices

$$n_1 < n_2 < n_3 < \cdots < n_k,$$

with corresponding terms

$$s_{n_1} \geq s_{n_2} \geq s_{n_3} \geq \cdots \geq s_{n_k}$$

Let n_{k+1} be the **smallest** index so that $n_{k+1} \geq n_k + 1$ ($\Rightarrow n_{k+1} > n_k$) and $s_{n_{k+1}} = \max\{s_n\}_{n=n_k+1}^{\infty}$. We get that $s_{n_{k+1}} \geq s_{n_k}$. So $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1}$ with $s_{n_1} \geq s_{n_2} \geq s_{n_3} \geq \cdots \geq s_{n_k} \geq s_{n_{k+1}}$.

By mathematical induction, we can choose an element s_{n_k} for every $k \in \mathbb{N}$, so that the subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ is monotone decreasing.

- **Case II.** At least one of these tails has no largest element, say

$$s^{(N)} = s_N, s_{N+1}, s_{N+2}, s_{N+3}, \ldots$$

Claim : If s_n is **any** element of this subsequence, then there exists $\bar{n} > n$ so that $s_{\bar{n}} > s_n$.

Suppose to the contrary that $s_{\bar{n}} \leq s_n, \forall \bar{n} > n$. Let $s = \max\{s_N, s_{N+1}, s_{N+2}, \ldots, s_n\}$. As $s \geq s_n \geq s_{\bar{n}}, \forall \bar{n} > n$. We see that $s = \max\{s^{(N)}\}$. It contradicts to the assumption that $s^{(N)} : s_N, s_{N+1}, s_{N+2}, s_{N+3}, \ldots$ has no maximum. This proves the claim.

Now, construct an increasing subsequence :

1. Let $n_1 = N$, so $s_{n_1} = s_N$
2. By the claim, there exists such index n_2 so that $n_1 < n_2$ and $s_{n_1} < s_{n_2}$.
3. In general, suppose we have chosen $n_1 < n_2 < \cdots < n_k$ with $s_{n_1} < s_{n_2} < \cdots < s_{n_k}$, we are able to find n_{k+1} so that $n_k < n_{k+1}$ and $s_{n_k} < s_{n_{k+1}}$.

So continuing this process, we obtain **a strictly increasing** subsequence $\{s_{n_k}\}_{k=1}^{\infty}$.

Both two cases show us that *we always find a monotone subsequence of any sequence.*

Theorem 3.48 (Bolzano-Weierstrass theorem for sequences)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence. By lemma 3.47 (page 103.), there exists a monotone subsequence $\{s_{n_k}\}_{k=1}^{\infty}$. Since $\{s_{n_k}\}_{k=1}^{\infty}$ is bounded, this subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ is also bounded. By theorem 3.42 (page 98.), this monotone bounded subsequence converges.

Corollary 3.49. *Every bounded sequence of real numbers has a Cauchy subsequence.*

Proof. It is a consequence from theorem 3.26 (page 88.).

The completeness property (or **completeness axiom** or **the least-upper-bound property**), *an ordered field F has the **the completeness property** if **every** nonempty subset S of F which is bounded above has **a least upper bound***, which has discussed in definition 2.2 (page 42.), is an essential ingredient of the proof of **Bolzano-Weierstrass theorem for sequences**. Bolzano-Weierstrass theorem is also an important part of the proof of theorem 3.26, *A sequence of **real***

numbers *converges if and only if it is Cauchy.* Indeed, one can prove that the completeness property 2.2 and theorem 3.26 are equivalent. In the general idea for considering the completeness of some abstract set, we sometimes consider its Cauchy sequences to figure out the completeness property.

Exercise

Prove or disprove the following statements.

1. Every sequence has an increasing subsequence.

2. Every sequence has a bounded subsequence.

3. Every unbounded sequence has a subsequence that diverges to ∞ or $-\infty$.

4. If a sequence has a greatest term, then every subsequence of the sequence has a greatest term.

5. Every unbounded sequence has no convergent subsequence.

Chapter 4

Limit and Continuity

4.1 Cluster Points and Isolated Points

Definition 4.1 (neighborhood and deleted neighborhood)**.** If $a \in \mathbb{R}$, then **ε-neighborhood of** a is the set

$$N_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$$
$$= (x - \varepsilon, x + \varepsilon)$$

A **deleted ε-neighborhood of** a is the set

$$N^*_\varepsilon(a) = \{x \in \mathbb{R} : 0 < |x - a| < \varepsilon\},$$

i.e. cut the center a of $N_\varepsilon(a)$.

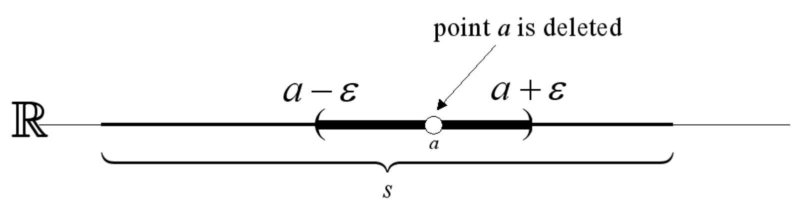


Figure 4.1: Deleted ε-Neighborhood $N^*_\varepsilon(a)$ is contained in S.

Definition 4.2 (cluster point)**.** Let $S \subseteq \mathbb{R}$. A point $a \in \mathbb{R}$ is called a **cluster point** or **accumulation point** or **limit point** of S provided that every deleted neighborhood $N_\varepsilon^*(a)$ contains at least one element of S.

Example 4.1. These are examples of cluster points and not cluster point of S.

1. By figure 4.1, all $x \in N_\varepsilon^*(a)$ is also in S. Then a is a cluster point of set S.
2. All points in $(0, 1)$ are cluster points of $(0, 1)$.
3. The points 0 and 1 are both cluster points of $(0, 1)$
4. All points in (a, b) and a and b are cluster points of (a, b).
5. All points in $[a, b]$ and are cluster points of $[a, b]$.
6. If x is not in $[a, b]$ then x is not a cluster point of $[a, b]$.
7. Let $S = (0, 1) \cup \{2\}$. Then 2 is **not** a cluster point of S.
8. Let $S = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$. $a = 0$ is an only one cluster point of S.

These examples show "a cluster point a of set S may be an element of S or may be not an element of S."

Definition 4.3. Let $S \in \mathbb{R}$. The set S' consisting the cluster points of S is called the **derived set of** S.

Example 4.2. These are examples of derived sets of set S.

1. $S = (0, 1) \Rightarrow S' = [0, 1]$
2. $S = (0, 1) \cup \{2\} \Rightarrow S' = [0, 1]$
3. $S = (a, b] \Rightarrow S' = [a, b], a < b$

4. Let $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \Rightarrow S' = \{0\}$.

5. We cannot find a cluster point of a set of integer $\mathbb{Z}$, then its derive set is empty set.

The next theorem makes it easy to find cluster points of set S.

Theorem 4.1. *Let $S \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then the following are equivalent.*

1. *a is a cluster point of S.*

2. *For every $\delta > 0$, there exists $x \in S$ with $0 < |x - a| < \delta$.*

3. *There exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $S \backslash \{a\}$ such that $x_n \to a$. (i.e. there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in S with $x_n \neq a, \forall n$ so that $x_n \to a$).*

4. *There exists a sequence $\{y_k\}_{k=1}^{\infty}$ with **distinct** terms of S (i.e. if $y_k = y_l \Rightarrow k = l$) such that $y_k \to a$.*

5. *Every neighborhood (nbhd) of a contains infinitely elements of S.*

Proof. We proof 1. → 2. → 3. → 4. → 5. → 1.)

(1. → 2.) Let a be cluster point of S. Then for every $\delta > 0$, by the definition, there exists at least one element of S in the deleted nbhd of $N_\varepsilon^*(a)$. Hence this x must satisfy $0 < |x - a| < \delta$.

(2. → 3.) Construct the sequence $\{x_n\}$ as follows. For each $n \in \mathbb{N}$, set $\delta_n = \frac{1}{n}$. By 3., there exists an element x_n of S s.t. $0 < |x_n - a| < \delta_n$, i.e.

$$a - \frac{1}{n} < x_n < a + \frac{1}{n}, \qquad x_n \neq a.$$

Let $n \to \infty$, by the squeeze theorem $x_n \to a$.

(3. $\to$ 4.) Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in S with $x_n \neq a$ and $x_n \to a$. Construct a sequence $\{y_k\}_{k=1}^{\infty}$ as follows. Set $n_1 = 1$ and $y_1 = x_{n_1} = x_1$. Set $\varepsilon = |x_1 - a|$ as $x_n \to a$ there exists $n_2 \in \mathbb{N}$ so that $|x_{n_2} - a| < \frac{\varepsilon}{2} = \frac{|x_1 - a|}{2}$. This makes sure that $x_2 \neq x_1$ and $x_2 \neq a$. Set $y_2 = x_{n_2}$.

Suppose we have chosen $y_1, y_2, \ldots, y_k$. So

$$|y_k - a| < \frac{1}{2}|y_{k-1} - a| < \frac{1}{2^2}|y_{k-2} - a| < \cdots < \frac{1}{2^{k-1}}|y_1 - a| = \frac{\varepsilon}{2^{k-1}}.$$

As $x_n \to a$, there exists $n_{k+1} \in \mathbb{N}$ so that $|x_{n_{k+1}} - a| < \frac{1}{2}|y_k - a|$. Set $y_{k+1} = x_{n_{k+1}}$. Then $|y_{k+1} - a| < \frac{1}{2}|y_k - a| < \frac{1}{2}\left(\frac{\varepsilon}{2^{k-1}}\right) = \frac{\varepsilon}{2^k}$ Continuing this way, we obtain element $y_1, y_2, y_3, \ldots$ of S so that

$$0 < |y_k - a| < \frac{\varepsilon}{2^{k-1}} \tag{4.1}$$

and $|y_k - a| \neq |y_l - a|$ if $k \neq l$.

Now by equation (4.1), and the squeeze theorem $|y_k - a| \to 0$ as $k \to \infty$. Then $y_k - a \to 0 \Rightarrow y_k \to a$.

(4. $\to$ 5.) Let $N_\varepsilon(a)$ be a nbhd of a. By 4., there exists a sequence $\{y_k\}_{k=1}^{\infty}$ of **distinct elements** of S such that $y_k \to a$. By the alternative definition of a limit, $N_\varepsilon(a)$ contain infinitely many terms of y_k since all these y_k are distinct. This shows that $N_\varepsilon(a)$ contains infinitely many (distinct) elements of S.

(5. $\to$ 1.) Let $N_\varepsilon(a)$ be any nbhd of a.

By 5. $N_\varepsilon(a)$ contains infinitely points of S. One of these elements may equal to a, but the infinitely remaining ones are different from A. So the deleted neighborhood $N_\varepsilon^*(a)$ contains even infinitely many elements of S. By the definition a must be a cluster point of S.

Theorem 4.2. *A finite set has no cluster points.*

Proof. Follows from property 4. of theorem 4.1, i.e. we cannot find $y_n \to a$ with y_k is different.

Definition 4.4 (isolated point)**.** Let $S \subseteq \mathbb{R}$. A point $a \in S$ which is not a cluster point of S is called an **isolated point** of S.

Remark. We may consider definition 4.4 of an isolated point a of S as

1. $a \in S$.
2. There exists a deleted nbhd $N_\varepsilon^*(a)$ which contains no element of S.

Example 4.3.

- Let $S = (0, 1) \cup \{2\}$. We can see that 2 is an isolated point of S.
- Let $S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$. Since $\frac{1}{n}$ is not a cluster point of $S \Rightarrow$ every element of S is an isolated point.
- Open and close intervals (a, b) and $[a, b]$ where $a < b$ have no isolated point.
- Let S be any nonempty finite set. By theorem 4.2, all elements in S are isolated points of S.
- A set of real numbers $\mathbb{R}$ has no isolated point.
- A set of rational numbers $\mathbb{Q}$ has no isolated point.
- A set of irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ has no isolated point.
- A set of integers $\mathbb{Z}$ contains all isolated points of itself.

Definition 4.5 (closure)**.** Let $S \subseteq \mathbb{R}$ and S' is derived set of set S. The **closure** of S is the set

$$\bar{S} = S \cup S',$$

i.e. the closure $\bar{S}$ of S contains elements of S and its cluster points.

Example 4.4.

- For $S = (0,1)$, its closure is $\bar{S} = [0,1]$
- For $S = (0,1) \cup \{2\} \Rightarrow \bar{S} = [0,1] \cup 2.$.
- $S = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \Rightarrow \bar{S} = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}.$
- The closure of a rational number set $\mathbb{Q}$ is a set of real numbers $\mathbb{R}$.
- Also an irrational number set $\mathbb{R}\backslash\mathbb{Q}$ has the same closure, a set of real numbers $\mathbb{R}$.
- The closure of an integer set $\mathbb{Z}$ is itself.

Theorem 4.3. *Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then $x \in \bar{S}$ if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $x_n \to x$.*

Proof. Exercise.

Theorem 4.4 (Blozano-Weierstrass theorem for set)**.** *Every bounded, infinite subset S of $\mathbb{R}$ has cluster point.*

Proof. Since S is infinite, there exists a countable subset $X \subseteq S$ whose elements we can list

$$X = \{x_1, x_2, \ldots, x_n, \ldots\}$$

So $\{x_n\}_{n=1}^{\infty}$ is a sequence of distinct point of S because S is bounded, the subset $X = \{x_n\}_{n=1}^{\infty}$ of S is also bounded. By the Bolzano-Weierstrass theorem for sequence 3.48 (page 105), there exists a convergent subsequence $\{y_k\}_{k=1}^{\infty} = \{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$.

Let $a = \lim_{n\to\infty} y_k$. Note that by theorem 4.1 property **??** (page **??**.), a is a cluster point of S.

Remark. We cannot omit the assumption that S must be bounded. By an example 4.2 part 5 (page 109), an integer set $\mathbb{Z}$ is infinite but it is unbounded, $\mathbb{Z}$ has no cluster point.

Exercise

1. Prove theorem 4.3 (page 112.).

2. Prove or disprove the following statements about sets of real numbers.

   (a) If $A \subseteq B$, then $A' \subseteq B'$

   (b) If $A' = B'$, then $A = B$

   (c) $A' \cup B' \subseteq (A \cup B)'$

   (d) $A' \cup B' \supseteq (A \cup B)'$

   (e) $A' \cap B' \subseteq (A \cap B)'$

   (f) $A' \cap B' \supseteq (A \cap B)'$

4.2 Some Topological Concepts

The topological concepts of **open** and **closed**, as applied to intervals, can be extended quite generally to sets. The concepts and definitions are based on **neighborhood** and **cluster point**.

A set whose elements are sets is often referred to as a **collection** of sets.

Definition 4.6 (union)**.** If $\mathcal{C}$ is a collection of sets, then the **union** of $\mathcal{C}$ is a set consisting of all elements that belong to *at least* one of the members of $\mathcal{C}$:

$$\begin{aligned} \cup\mathcal{C} &= \cup\{S : S \in \mathcal{C}\} \\ &= \{x : x \in S \text{ for } \textbf{some } S \in \mathcal{C}\} \end{aligned}$$

Definition 4.7 (intersection)**.** The **intersection** of collection $\mathcal{C}$ is a set consisting of those elements belonging to *every* member of $\mathcal{C}$:

$$\begin{aligned} \cap\mathcal{C} &= \cap\{S : S \in \mathcal{C}\} \\ &= \{x : x \in S \text{ for } \textbf{all } S \in \mathcal{C}\} \end{aligned}$$

Example 4.5.

- $\cup\{N_\delta(a) : \delta > 0\} = \mathbb{R}$
- $\cap\{N_\delta(a) : \delta > 0\} = \{a\}$
- $\cup\{[n-1, n] : n \in \mathbb{Z}\} = [0, \infty)$
- $\cap\left\{\left[0, \frac{1}{n}\right] : n \in \mathbb{Z}\right\} = \{0\}$
- $\cup\left\{\left(0, \frac{1}{n}\right) : n \in \mathbb{Z}\right\} = (0, 1)$
- $\cap\left\{\left(0, \frac{1}{n}\right) : n \in \mathbb{Z}\right\} = \phi$
- $\cap\left\{\left(-\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{Z}\right\} = \{0\}$

Definition 4.8 (open set)**.** A set S of real numbers is **open in** $\mathbb{R}$ (briefly, **open**) if, for every $x \in S$, there exists a positive real number $\delta > 0$ such that δ-neighborhood $N_\delta(x)$ is contained in S, i.e. $N_\delta(x) \subseteq S$.

The real number δ depends on the point x, so different values of δ might be chosen for different point x of S.

Example 4.6.

- **Claim** : An open interval $(a, b), a < b$ is open in $\mathbb{R}$.

  Consider for any $x \in (a, b)$. Let $\delta = \min(x - a, b - x)$ thus $N_\delta(x) \subseteq (a, b)$.

- **Claim** : An half open interval $(a, b], a < b$ is **not** open in $\mathbb{R}$.

  Since $b \in (a, b]$ but $N_\delta(b)$ are not subsets of $(a, b]$ for all $\delta > 0$.

- Any close interval $[a, b], a < b$ is **not** open.

- A set of real numbers $\mathbb{R}$ or $(-\infty, \infty)$ is open.

- The empty set ϕ is open. Since ϕ has no elements whatever, there are no elements of ϕ that fail to have a neighborhood contained in ϕ.

- A **singleton** (a set which has only one element) is **not** open.

- A set of integers $\mathbb{Z}$ is **not** open. (Exercise)

- A nonempty finite set is **not** open. (Exercise)

Theorem 4.5. *The union of any collection of open sets is open.*

Proof. Let $\mathcal{C}$ be any collection of open sets. Let $\mathsf{C} = \cup\mathcal{C}$ and $x \in \mathsf{C}$ is arbitrary. Since $x \in \mathsf{C}$ then x must belong to at least one element of $\mathcal{C}$, says G. If $G \in \mathcal{C} \Rightarrow G \subseteq \cup\mathcal{C} = \mathsf{C}$. Since G is open then there exists $\delta > 0$ so that $N_\delta(x) \subseteq G \subseteq \mathsf{C}$. By the arbitrariness of x, we can conclude that C or the union of open sets is also open.

Example 4.7. Consider the collection $\mathcal{C} = \{(i, i+1) : i \in \mathbb{Z}\}$. By theorem 4.5, $\cup\mathcal{C}$ or $\mathbb{R}\backslash\mathbb{Z}$ is open.

Theorem 4.6. *The intersection of a finite collection of open sets is open.*

Proof. Let $\mathcal{C}$ be a finite collection of open sets, say $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$. To say that x belongs to the intersection means that x belongs to each $C_i, i = 1, 2, \ldots, n$. Since all of the sets C_i are open, there exist positive real numbers $\delta_i > 0$ corresponding to set C_i such that $N_{\delta_i}(x) \subseteq C_i$. Let $\delta = \min(\delta_1, \delta_2, \ldots, \delta_n)$. So N_δ is contained in **every** $C_i \Rightarrow N_\delta \subseteq C_1 \cap C_2 \cap \cdots \cap C_n$.

Remark. The intersection of an **infinite** collection of open sets is **not necessary** to be open. We can see from an example

$$\bigcap\left\{\left(-\frac{1}{n},\frac{1}{n}\right):n\in\mathbb{Z}\right\}=\{0\},$$

where $\{0\}$ is a singleton which is not open.

Theorem 4.7. *A set of real numbers is open in* $\mathbb{R}$ *if and only if it is the union of a collection of open intervals.*

Proof. ($\Rightarrow$) Let G be open. Then for each $x \in G$, there is $\delta_x > 0$ such that $N_{\delta_x}(x) \subseteq G$. It follows that G is the union f all these open interval $N_{\delta_x}(x)$.

($\Leftarrow$) It follows immediately from theorem 4.5 (page 115.).

Definition 4.9 (closed set)**.** A set S of real numbers is **closed in** $\mathbb{R}$ (briefly, **closed**) if every cluster point[1] belongs to S.

Example 4.8.

- A closed interval $[a, b], a < b$ is closed in $\mathbb{R}$.
- An open interval $(a, b), a < b$ is not closed in $\mathbb{R}$ since a and b are cluster points of (a, b) but there are not contained in (a, b)
- A set of real numbers $\mathbb{R}$ or $(-\infty, \infty)$ is closed.
- The empty set ϕ is closed. Since ϕ has no elements whatever, there are no elements of ϕ that fail to have a cluster point contained in ϕ.
- A **singleton** and the set of integer sets $\mathbb{Z}$ are closed since they have no cluster point.
- A nonempty finite set is closed.

[1]See the definition of a cluster point in definition 4.2 (page 108.)

- The set of rational numbers $\mathbb{Q}$ is not closed since irrational numbers are its cluster points but they are not contained in $\mathbb{Q}$.

Remark. Set of real numbers $\mathbb{R}$ and empty set ϕ are **both open and closed** subset of $\mathbb{R}$. (Why?)

Theorem 4.8. *Any subset of real numbers is closed if and only if its complement is open.*

Proof. ($\Rightarrow$) Suppose $S \subseteq \mathbb{R}$ is closed and consider its complement $S^c = \mathbb{R} \backslash S$. For any $x \in S^c, x \notin S$ thus x is not a cluster point of $S \Rightarrow$ for $\delta > 0$, $N_\delta(x) \cap S = \phi$. This shows that $N_\delta(x) \nsubseteq S \Rightarrow N_\delta(x) \subseteq S^c$. By an arbitrariness of $x \in S^c$, S^c contains a neighborhood of x shows S^c is open.

($\Leftarrow$) Conversely, suppose S^c is open. As S^c is open, for any $x \in S^c$ there exists $\delta > 0$ so that $N_\delta(x) \subseteq S^c$. If is impossible to have $N_\delta(x) \cap S \neq \phi$ for all $\delta > 0$. Hence, for $x \in S^c$, x cannot be a cluster point of S. It follows that if there exists a cluster point of S, all cluster points must belong to S. Set S is therefore closed.

Corollary 4.9. *Any subset of real numbers is open if and only if its complement is closed.*

Proof. Exercise.

Theorem 4.10. *The intersection of any collection of closed set is closed.*

Proof. We use that fact that the complement of a union is the intersection of the complements. Suppose $\mathcal{C}$ is a collection of a number of closed set. Consider the complement of the intersection of collection $\mathcal{C}$, $(\cap\mathcal{C})^c$. By De Morgan's Law,

$$(\cap\mathcal{C})^c = \cup\mathcal{G},$$

where $\mathcal{G}$ is a collection of all complements of the sets in $\mathcal{C}$. By theorem 4.8, the elements of $\mathcal{G}$ are open. So $\cup\mathcal{G}$ is open (theorem 4.5). By theorem 4.8 again, the complement of $\cap\mathcal{C}$ is open then $\cap\mathcal{C}$ is closed.

Theorem 4.11. *The union of a finite collection of closed sets is closed.*

Proof. Exercise.

Theorem 4.12. *A subset S of $\mathbb{R}$ is closed if and only if every convergent sequence of points in S converges to a point in S.*

Proof. ($\Rightarrow$) Let S be closed in $\mathbb{R}$ and let $\{x_n\}$ be a convergent sequence which $x_n \in S, \forall n$ and $x_n \to x$. Suppose to the contrary that $x \notin S \Rightarrow x \in S^c$. By theorem 4.8, S^c is open and there exists $\delta > 0, N_\delta(x) \subseteq S^c$. However $x_n \in S, \forall n$ then $x_n \notin N_\delta(x)$. It contradicts to $x_n \to x$. This shows that every convergent sequence of elements in S converges to the element in the same set.

($\Leftarrow$) Let S be a subset of $\mathbb{R}$. Suppose x is a cluster point of S. By theorem 4.1 part 3, there exists a convergent sequence $\{x_n\}$, $x_n \in S, x_n \neq x$ and $x_n \to x$. By the assumption that every convergent sequence of points in S converges to a point in S then x must be in S. Since S contains its all cluster points then S is closed.

Exercise

1. Show that set of real numbers $\mathbb{R}$ and empty set ϕ are **both open and closed** subset of $\mathbb{R}$.

2. Let S be a subset of $\mathbb{R}$, if S is both open and closed, determine what the set S is. Give the reason.

3. Prove corollary 4.9.

4. Prove theorem 4.11.

5. Determine whether the following sets are open, closed, or neither.

(a) $\displaystyle\bigcap_{n=1}^{\infty}\left(0,\frac{1}{n}\right)$

(b) $\displaystyle\bigcap_{n=1}^{\infty}\left[0,\frac{1}{n}\right]$

(c) $\displaystyle\bigcup_{n=1}^{\infty}\left[0,\frac{1}{n}\right]$

(d) $\displaystyle\bigcup_{n=1}^{\infty}(n,n+1)$

(e) $\displaystyle\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n},3+\frac{1}{n}\right)$

(f) $\displaystyle\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1-\frac{1}{n}\right]$

(g) $\displaystyle\bigcup_{n=1}^{\infty}[n,n+1)$

(h) $\displaystyle\bigcap_{n=1}^{\infty}[n,\infty)$

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