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Abstract Collection

Integrable geodesic flows with rational first integrals

S. V. Agapov[1,2], *V. V. Shubin*[3]

[1]*Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia*;
[2]*Novosibirsk State University, Novosibirsk, Russia*; agapov.sergey.v@gmail.com
[3]*Luxoft d.o.o Beograd, Belgrade, Serbia*; vlad.v.shubin@gmail.com

We study integrable geodesic flows on 2-surfaces. In most known examples, the additional integrals are polynomial in momenta. Polynomial integrals of small degrees are well-studied and classified in both local and global aspect of the problem.

On the other hand, as proved in [1], there exist local Riemannian metrics on 2-surfaces with integrable geodesic flows such that additional integrals are rational in momenta with any given degrees of a numerator and a denominator. However, constructing such examples in an explicit form turned out to be a very difficult problem. In this talk we will describe various methods and approaches which allowed us to construct rich families of 2-dimensional Riemannian metrics with integrable geodesic flows admitting additional rational first integrals (see [2], [3]).

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Collision of shock waves in plasma medium: An analytical study

Sherin Agnus[1], *Rajeswari Seshadri*[2]

Department of Mathematics, Pondicherry University, Kalapet 605014, India;
[1]agnus.1112@pondiuni.ac.in, [2]seshadrirajeswari@pondiuni.ac.in

The collision of shock waves in plasma medium is modeled as a coupled Ramani equation. In this paper, we investigate the coupled Ramani equation using two main analytical techniques, namely Lie symmetry analysis and the singular manifold method. The application of Lie group analysis to the governing equation yields four-dimensional Lie algebra. Then we construct an one-dimensional optimal system of subalegbras to study the classification of group-invariant solutions. The similarity reduction for each symmetry in the optimal system transforms the coupled Ramani equation into a system of ordinary differential equations (ODEs). This includes certain directly solvable ODEs and a traveling-wave reduction. For one of the symmetry-reduced ODEs that cannot be solved directly, the singular manifold method which is a truncated expansion analysis is used to investigate the solution and examine the underlying properties.

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Symmetries and exact solutions of a Non-standard Diffusion SIR Models

Maryam Ahmadpoortorkamani[1], *Alexei Cheviakov* [2]

[1]*Department of Mathematics and Statistics, University of Saskatchewan, Canada*;
[1]qzm279@usask.ca, [2]shevyakov@math.usask.ca

A novel Susceptible-Infected-Recovered (SIR) epidemiological partial differential equation (PDE) system developed by Vaziry, Kolokolnikov, and Kevrekidis (Royal Society Open Science, 9(10), 220064, 2022) involves a nonstandard spatial diffusion term that models infection spread through commuting infected individuals. We present a symmetry classification of the PDE family and calculate some ordinary differential equation (ODE) or ODE system reductions. Resulting self-similar solutions are computed by a combination of analytical and numerical techniques. These solutions satisfy a simple boundary value problem and model an incoming infection wave.

A novel approximate analytical approach for the delay differential equations with fractional Caputo-Fabrizio derivative and application to economic models

S. Moyo[1,3], *M. O. Aibinu*[2,3]

[1]*Department of Applied Mathematics and School for Data Science and Computational Thinking, Stellenbosch University, South Africa*; smoyo@sun.ac.za

[2]*Institute for Systems Science, Durban University of Technology, Durban 4000, South Africa*; moaibinu@yahoo.com

[3]*National Institute for Theoretical and Computational Sciences (NITheCS), South Africa*;

Fractional differential equations are indispensable in the modelling of several physical problems with precision. Situations often arise that show discrepancy between the experimental results and what is obtained as the results from the models that are given by the differential equations with integer-order. Therefore, fractional differential equations have attracted attention as a powerful tool in the modelling of physical phenomena with precision. Efforts on the solutions of fractional differential equations is a vast research area. This study considers using an approximate analytical approach, which is a hybrid of Sumudu transform for solving differential equations with Caputo-Fabrizio derivative and time delay. Furthermore, the study considers introducing the Caputo-Fabrizio derivative and time delay into an economic model and examines using a hybrid of Sumudu transform to obtain its solution.

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Symmetries and Equivalence of Systems of Shallow Water Equations Over Horizontal and Inclined Bottom

A. V. Aksenov[1]

[1]*Lomonosov Moscow State University, Moscow, Russia*; aksenov@mech.math.msu.su

In dimensionless variables, the system of equations of two-dimensional shallow water over a horizontal bottom has the following form [1]

$$\begin{aligned} &u_t + uu_x + vu_y + \eta_x = 0\,,\\ &v_t + uv_x + vv_y + \eta_y = 0\,,\\ &\eta_t + \big(u(\eta+h)\big)_x + \big(v(\eta+h)\big)_y = 0\,. \end{aligned} \tag{1}$$

Here $u = u(x, y, t)$, $v = v(x, y, t)$ are components of the depth-averaged horizontal velocity; $\eta = \eta(x, y, t)$ is free surface elevation; $\eta + h$, $\eta + h \geq 0$, $h = const$ is depth. Let us write the system of equations of two-dimensional shallow water over an inclined bottom in the following form

$$\begin{aligned} &u'_{t'} + u'u'_{x'} + v'u'_{y'} + \eta'_{x'} = 0\,,\\ &v'_{t'} + u'v'_{x'} + v'v'_{y'} + \eta'_{y'} = 0\,,\\ &\eta'_{t'} + \big(u'(\eta' + ax' + by')\big)_{x'} + \big(v'(\eta' + ax' + by')\big)_{y'} = 0\,. \end{aligned} \tag{2}$$

Here $u' = u'(x', y', t')$, $v' = v'(x', y', t')$ are components of the depth-averaged horizontal velocity; $\eta' = \eta'(x', y', t')$ is free surface elevation; $\eta' + ax' + by' \geq 0$, $a, b = const$ is depth.

Using the algorithm [2], the symmetries of the systems of equations (1) and (2) were found. The Lie algebras of the symmetry operators are finite-dimensional, so these systems of equations cannot be linearised by the point transformation. One-dimensional systems of shallow water equations are linearised by the point transformation: the system of shallow water equations over a horizontal bottom is linearised by the hodograph transformation [1], and the system of shallow water equations over an inclined bottom is linearised by the Carrier–Greenspan transformation [3] (see also [4, 5]).

Proposition. *Point transformation*

$$\begin{aligned} &x' = x + \frac{at^2}{2}\,, \quad y' = y + \frac{bt^2}{2}\,, \quad t' = t\,,\\ &u' = u + at\,, \quad v' = v + bt\,, \quad \eta' = \eta - a\Big(x + \frac{at^2}{2}\Big) - b\Big(y + \frac{bt^2}{2}\Big) + h \end{aligned} \tag{3}$$

determines the equivalence of the systems of equations (1) *and* (2).

Remark. The transformation inverse to the transformation (3) has the form

$$\begin{aligned} &x = x' - \frac{at'^2}{2}\,, \qquad y = y' - \frac{bt'^2}{2}\,, \qquad t = t'\,,\\ &u = u' - at'\,, \qquad v = v' - bt'\,, \qquad \eta = \eta' + ax' + by' - h\,. \end{aligned}$$

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Nonlinear Boussinesq Type Equations Under Mixed Boundary Conditions For Temperature

G. V. Alekseev[1,2], *O. V. Soboleva*[1,2]

[1] *Institute of Applied Mathematics FEB RAS, Vladivostok, Russia*; alekseev@iam.dvo.ru (G.A.), soboleva22@mail.ru (O.S.)

[2] *Department of Mathematical and Computer Modelling, Far Eastern Federal University, Vladivostok, Russia*; alekseev@iam.dvo.ru (G.A.), soboleva22@mail.ru (O.S.)

In 1959 L.V. Ovsyannikov published the paper [1] on group properties of differential equations of nonlinear thermal conductivity. After publication of this paper, the direction related to the study of the qualitative properties of solutions of heat and mass transfer equations with coefficients dependent on temperature and/or concentration of dissolved substance began to develop (see, for example, [2, 3, 4]). But theoretical questions related to the study of correctness of the respective boundary value problems were studied to a much lesser extent.

The purpose of this work is to analyze the global solvability and local uniqueness of solution of the boundary value problem for the generalized Boussinesq heat transfer model having the form

$$-\mathrm{div}(\nu(T)\nabla\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = b(T)T\mathbf{G},\ \mathrm{div}\mathbf{u} = 0 \text{ in } \Omega, \tag{1}$$

$$-\mathrm{div}(\lambda(T)\nabla T) + (\mathbf{u}\cdot\nabla)T = 0 \text{ in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{g} \text{ on } \Gamma,\ \lambda(T)(\partial T/\partial\mathbf{n} + \alpha(T)T) = \chi \ \text{ on } \Gamma_N \text{ and } T = \psi \text{ on } \Gamma_D. \tag{3}$$

Here $\mathbf{u}$ is the velocity vector, T is the temperature of medium, $p = P/\rho_0$, where P is the pressure, $\rho_0 = \mathrm{const}$ is the fluid density, $\nu = \nu(T) > 0$ is the kinematic (molecular) viscosity coefficiemt, $\lambda = \lambda(T) > 0$ is the thermal conductivity coefficient, $b \equiv b(T)$ is the heat expansion factor, $\alpha = \alpha(T)$ is the heat exchange coefficient, $\mathbf{G} = -(0, 0, G)$ is the gravitational acceleration.

We develop mathematical apparatus of studying inhomogeneous boundary value problem under consideration. It is based on using of a weak solution of the boundary value problem (1)–(3) and construction of liftings of the inhomogeneous boundary data (3). They remove the inhomogeneity of the boundary data and reduce original problem (1)–(3) to equivalent homogeneous boundary value problem. Based on this apparatus we will prove the theorem on the global existence of a weak solution to boundary value problem under study and establish important properties of the solution. In particular, we will prove the validity of the maximum principle for the temperature. We will also establish sufficient conditions for the data, ensuring the local uniqueness of the weak solution having an additional property of smoothness for temperature.

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Exact solitary wave solutions for a coupled gKdV-NLS system

S.C. Anco

Department of Mathematics and Statistics, Brock University, Canada; sanco@brocku.ca

A coupled gKdV-NLS system

$$u_t + \alpha u^p u_x + \beta u_{xxx} = \gamma(|\psi|^2)_x,$$
$$i\psi_t + \kappa\psi_{xx} = \sigma u\psi$$

with a general nonlinearity power $p > 0$ is studied. This system has been introduced in the literature [1, 2] to model energy transport in anharmonic crystal materials.

There is a strong interest in obtaining exact solutions describing frequency-modulated solitary waves

$$u = U(x - ct), \quad \psi = e^{i\omega t}\Psi(x - ct),$$

where c is the wave speed, and ω is the modulation frequency. For the KdV case $p = 1$, some solutions have been found in [1], while for the mKdV case $p = 2$ in [2], no exact solutions were found. Nothing has been done for higher nonlinearities $p \geq 3$.

In the present work, we derive exact solutions for $p = 1, 2, 3, 4$, starting from the travelling wave ODE system satisfied by U and Ψ. The method is new:
(i) obtain first integrals by use of multi-reduction symmetry theory [3];
(ii) apply a hodograph transformation which leads to triangular (decoupled) system;
(iii) introduce an ansatz for polynomial solutions of the base ODE;
(iv) characterize conditions under which solutions yield solitary waves;
(v) solve an algebraic system for the coefficients in the ansatz under those conditions.

The resulting solitary waves exhibit a wide range of features: bright and dark peaks; single peaked and multi-peaked; zero and non-zero backgrounds.

This method can be applied more generally to nonlinear systems of dispersive wave equations.

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Equivalence Transformations and Solvable Forms of Nonlinear Schrödinger Equations with Variable Coefficients: An Application of Lie's Invariance Criterion

M. Zainab[1], *A. Aslam*[2], *M. Safdar*[3]

[1]*School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), Islamabad-44000, Pakistan* ; marya.zainab@sns.nust.edu.pk

[2] *School of Electrical Engineering and Computer Science (SEECS), National University of Sciences and Technology (NUST), Islamabad-44000, Pakistan* ; adnan.aslam@seecs.edu.pk

[3]*School of Mechanical and Manufacturing Engineering (SMME), National University of Sciences and Technology (NUST), Islamabad-44000, Pakistan* ; msafdar@smme.nust.edu.pk

This study employs Lie's invariance infinitesimal criterion to derive continuous equivalence transformations for a class of nonlinear Schrödinger equations characterized by variable coefficients. Within the framework of these transformations, we formulate differential invariants which play a crucial role in exposing solvable forms of the considered Schrödinger equations. The deduced invariants lead to remarkable reductions of the Schrödinger equations compared to the general nonlinear Schrödinger equations under consideration. Such reductions are shown to facilitate the derivation of exact solutions for the Schrödinger equations.

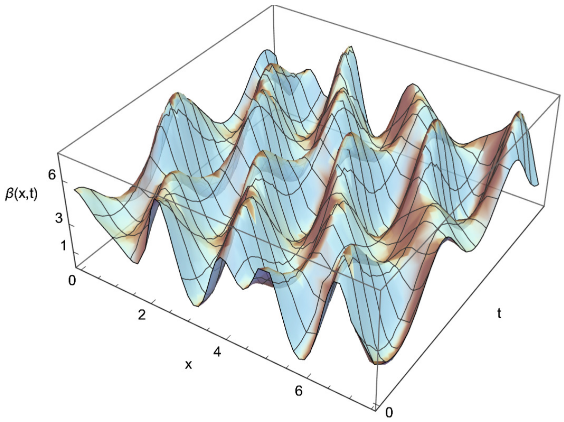
Reality conditions for the KdV equation and quasi-periodic solutions in finite phase spaces

J. Bernatska

Department of Mathematics, University of Connecticut; jbernatska@gmail.com

In this paper reality conditions for quasi-periodic finite-gap solutions of the KdV equation are determined. That is, real-valued quasi-periodic solutions, which describe nonlinear waves, are proposed in every finite phase space of the hierarchy of hamiltonian systems of the KdV equation. These solutions are expressed in terms of the abelian $\wp_{1,1}$-function, which comes as a result of algebro-geometric integration.

An effective computation of such solutions is suggested, and illustrated in genera 2 and 3. In particular, this wave arises in a 6-dimensional (3-gap) phase space.



The talk is based on preprint arXiv:2312.10859.

Reduction Operators for Monge-Ampère Equations

N. Bîlă

Department of Mathematics and Computer Science, Fayetteville State University, North Carolina, United States; nbila@uncfsu.edu

In this talk, reduction operators related to two-dimensional Monge-Ampère equations are discussed. A degenerated case that occurs while applying the nonclassical method (due to Bluman and Cole) to these types of nonlinear partial differential equations is studied. It is shown that specific Monge-Ampère equations may be reduced to systems of first order partial differential equations, and, additionally, their solutions are related to Monge and Bateman equations. The connection of these results with the direct method (by Clarkson and Kruskal) is also presented.

Toward the existence of nonlinear solitary waves (solitons) in a freely interacting transonic viscous flow. Hypotheses, degeneracy and regularization of the model

Andrey Nikolaevich Bogdanov[1,2] *and Igor Mikhailovich Kondratyev*[2,3]

[1]*Institute of Mechanics, Lomonosov Moscow State University, 119192 Moscow, Russia*;
bogdanov@imec.msu.ru
[2]*Bauman Moscow State Technical University, 105005 Moscow, Russia*;
[3]*Mechanical Engineering Research Institute of the Russian Academy of Sciences, 101990 Moscow, Russia*;

The necessity of regularizing the classical model for studying the propagation of solitons in a nonstationary transonic flow over a solid surface in the mode of free viscous-inviscid interaction is substantiated. Keywords: gas dynamics, nonlinear waves, solitons, transonic flow, viscous-inviscid interaction, boundary layer.

The hypothesis of the existence of stable solitary waves (solitons) in a freely viscous-inviscid interacting flow was expressed by V.I. Zhuk [1]. For the existence of waves of this kind, it is necessary to combine the action of a wave of nonlinearity increasing the steepness of the profile and, conversely, stretching the dispersion profile. The wave dissipation suppressing the wave process, reducing the amplitude of the waves, will not prevent the tipping of the wave front. Understanding the behavior of solitons in the boundary layer plays an important role in connection with the concept [2] of the soliton nature of the so-called K-mode of laminar flow destruction. Namely, the "spike bursts" (of negative amplitude) observed on oscillograms are not stochastic in nature, but represent definitely formed periodic structures with constant properties (the distances of maintaining shape constancy during the propagation of disturbances downstream amount to dozens of their characteristic sizes and more than a hundred boundary layer thicknesses).

At large longitudinal gradients of flow parameters, boundary layers exert, through a pressure gradient, a reverse effect on the main flow, in interaction with which they arise, and thus are processes with self-induced pressure. To distinguish from the usual (classical) Prandtl boundary layers, such boundary layers are called non-classical.

The main achievements in the analytical study of such non-classical boundary layers were obtained using the "three-deck" model [3]. Perturbations of the flow field during the propagation of soliton-like waves in the boundary layer have an amplitude exceeding the permissible by "three-deck" model and are studied using the "four-deck" model [4].

Further development of the asymptotic theory for boundary layers of this kind revealed that the description of the unsteady free visco-inviscid interaction in the transonic regime using three- and four-deck asymptotic models gives an incomplete picture of the flow field, since the Lin-Reissner-Qian equation included in the model turned out to be degenerate hyperbolic, in connection with which regularization of the three-deck model was proposed [5]. The regularized model makes it possible to describe the processes of wave propagation in all directions in the flow field, and not only upstream, and to analyze their development, which was previously excluded from consideration while using the classical model. Thus, there are grounds to assume that the study conducted in [1] does not provide a complete picture of the processes of behavior of solitons in the transonic viscous flows.

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Lie Group Geometry in the Group Analysis of the One-Dimensional Kinetic Equation

A. V. Borovskikh

Mech&Math Department, Lomonosov Moscow State University, Moscow, Russia; Scientific and Educational Mathematical Center of North Ossetian State University named after. K.L. Khetagurov, Vladikavkaz, Russia; bor.bor@mail.ru

Let G be an n-dimensional group parameterised by local coordinates $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ and Ξ be the corresponding generating algebra with the basis $\{\Xi_\alpha = \xi^i_\alpha(x)\partial_i\}^n_{\alpha=1}$.

Theorem 1. *The set of metrics $g_{ij}(x)dx^i dx^j$, defined on G and remaining invariant under the action of this group, forms a linear space of dimension $\frac{n(n+1)}{2}$.*

Theorem 2. *There are exactly n linear differential forms $\omega^\alpha = \omega^\alpha_i dx^i$ $(\alpha = 1, \dots, n)$ that are invariant under algebra Ξ.*

Corollary. *All metric forms from Theorem 1 are quadratic forms $ds^2 = q_{\alpha\beta}\omega^\alpha\omega^\beta$ of ω^α with constant coefficients $q_{\alpha\beta}$.*

Let us denote by ω^i_α the matrix inverse to the matrix ω^α_i of coefficients of differential forms ω^α. Then all operators $\Omega_\alpha = \omega^i_\alpha\partial_i$ commute with all Ξ_β: $[\Omega_\alpha, \Xi_\beta] = 0$.

Definition. *We call the algebra Ω formed by solutions of the system of equations $[\Xi_\alpha, \Omega] = 0$ dual to the algebra Ξ.*

Let

$$\frac{dx^1}{\phi^1(x)} = \dots = \frac{dx^n}{\phi^n(x)} \tag{1}$$

be equations of a family of curves invariant under the group G. Multiplying (1) by a suitable factor, it is possible to obtain $\omega^\alpha_i\phi^i$ be constants. Denote them by λ^α.

For any invariant metric $ds^2 = q_{\alpha\beta}\omega^\alpha\omega^\beta$ the unit tangent vector $\tau = (\tau^i)$ to curve (1) has the form $\tau^i = \frac{dx^i}{ds} = \frac{\phi^i(x)}{\sqrt{Q}}$, $Q = q_{\alpha\beta}\lambda^\alpha\lambda^\beta$ and therefore on the tangent vector τ the corresponding linear forms turn out to be constant: $\omega^\alpha_i\tau^i = \frac{\lambda^\alpha}{\sqrt{Q}}$.

Remark. The vector $\phi^i = \lambda^\alpha\omega^i_\alpha$ is a linear combination of vector fields generating the dual algebra Ω. Therefore the invariant under the algebra Ξ trajectories (1) are in fact trajectories of a one-parameter subgroup from the dual algebra.

Let us denote by $\tau_N = (\tau^k_N)$ the Frenet frame, numbered by the index $N = 1, \dots, n$, and, accordingly, $\lambda^\alpha_N = \omega^\alpha_i\tau^i_N$.

Theorem 3. *In terms of the quantities λ^α_N, the Frenet system has the form*

$$\frac{d\lambda^\gamma_N}{ds} + H^\gamma_{\alpha\beta}\lambda^\alpha_N\lambda^\beta_1 = -\varkappa_{N-1}\lambda^\gamma_{N-1} + \varkappa_N\lambda^\gamma_{N+1},$$

where $H^\gamma_{\alpha\beta}$ are constants defined by the formula

$$H^\gamma_{\alpha\beta} = -\frac{1}{2}[q^{\gamma\sigma}q_{\mu\beta}C^{*\mu}_{\alpha\sigma} + C^{*\gamma}_{\alpha\beta} + q^{\gamma\sigma}q_{\alpha\nu}C^{*\nu}_{\beta\sigma}] \tag{2}$$

and $C^{*\gamma}_{\alpha\beta}$ are structure constants of the algebra with basis Ω_α.

Corollary. *For all invariant curves all their curvatures and all quantities λ^α_N are constant.*

Theorem 4. *For any metric $g_{ij}(x) = q_{\alpha\beta}\omega^\alpha$ the corresponding Riemann and Ricci tensors have the form $R^l_{ijk} = M^\theta_{\alpha\beta\gamma}\omega^l_\theta\omega^\alpha_i\omega^\beta_j\omega^\gamma_k$, $R_{ij} = M_{\alpha\beta}\omega^\alpha_i\omega^\beta_j$, where $M^\theta_{\alpha\beta\gamma} = -C^{*\sigma}_{\beta\gamma}H^\theta_{\alpha\sigma} + H^\theta_{\sigma\beta}H^\sigma_{\alpha\gamma} - H^\theta_{\sigma\gamma}H^\sigma_{\alpha\beta}$, $M_{\alpha\beta} = M^\gamma_{\alpha\beta\gamma}$, $H^\sigma_{\alpha\beta}$ are constants defined by the formula (2), and $C^{*\gamma}_{\alpha\beta}$ are structure constants of the algebra Ω.*

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Equilibrium model of a mixing layer in a stratified fluid: application to deep-sea currents and internal hydraulic jumps

A. A. Chesnokov[1], *V. Yu. Liapidevskii*[2]

Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia;
[1]chesnokov@hydro.nsc.ru, [2]liapid@hydro.nsc.ru

Gravity currents are ubiquitous in geophysical, industrial, and environmental settings. Mixing, often driven and dominated by shear instability, plays a crucial role in the dynamics of these currents. Shear flows of stratified fluid over an uneven bottom are widespread in nature. The results of recent field observations of deep-sea currents in the Atlantic Ocean (Vema channel, Romanche and Chain fracture zones) are presented in [1, 2]. It has been established that it is possible to distinguish an active bottom layer, consisting of a homogeneous "flow core" and an intermediate layer in which the flow interfaces with the upper "passive" layer. Such a layered flow pattern and ideal channel geometry, ensuring two-dimensionality of the flow, are assumed when constructing a mathematical model. In our works [3, 4], using the Boussinesq approximation, a model was obtained that describes internal hydraulic jumps and mixing of homogeneous co-directional flows of different densities.

In this work, we focus on three-layer stratified flow, taking into account the entrainment of fluid from the outer layers into the intermediate vortex layer. The equations of motion are presented in the form of a non-linear system of inhomogeneous conservation laws. It is assumed that the rate of fluid entrainment into the vortex layer is specified by the equilibrium condition within the framework of a more general three-layer model [4]. This assumption allows us to significantly simplify the model and present it in the form of an evolutionary system of four equations. Using the characteristic velocities of the proposed model, we define the concept of a supercritical (subcritical) three-layer flow. We study classes of stationary flows over an uneven bottom and construct examples of continuous and discontinuous solutions that describe the spatial evolution of the mixing layer. It is shown that the model is applicable to describe the characteristic features of mixing and flow splitting in deep-sea currents [1, 2]. We perform unsteady calculations of the stratified flow over a combined obstacle and show that the formation of an internal hydraulic jump leads an intense mixing and growth of the vortex intermediate layer. The possibility of controlling the position of an internal hydraulic jump on the leeward side of an obstacle is shown. Experimental data [5] confirm the results of numerical simulation.

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Submodels of 2-d model of porous medium with an external non-stationary source or absorption

Yu. A. Chirkunov and M. Yu. Chirkunov

Department of Information and Economic Systems, Reshetnev Siberian State University of Science and Technology, Krasnoyarsk, Russia; chr101@mail.ru, mihail0807@mail.ru

For a three-dimensional generalized model of nonlinear hydroacoustics by Khokhlov-Zabolotskaya-Kuznetsov in a cubic nonlinear medium in the absence of dissipation, the attenuation of ultrasonic beams after the formation of shock fronts is studied. Earlier, in the work of one of the authors for this model, submodels were obtained and studied, described by non-stationary solutions, invariant with respect to some three-dimensional subgroups of the main ten-parameter group of the differential equation that defines this model. In our work, four non-stationary submodels of this model that are invariant with respect to four-parameter subgroups of the main group of this differential equation are obtained and studied. These submodels are new submodels and have not been previously noted in the literature. They are given by invariant solutions of rank 1. Among these 4 submodels, 2 submodels describe axisymmetric ultrasonic beams, the remaining 2 describe one-dimensional ultrasonic beams. The search for invariant solutions that define these 4 submodels is reduced to solving of nonlinear integral equations, the implicit solutions of which are obtained in the form of nonlinear algebraic equations containing transcendental functions. These submodels are used to study the propagation of ultrasonic beams, for which either the acoustic pressure and its rate of change or the acoustic pressure and its gradient are given at the initial time at a fixed point. Conditions are obtained that ensure the existence and uniqueness of solutions to boundary value problems describing these processes. This makes it possible to correctly carry out numerical calculations in the study of these processes. As a result of the numerical solution of these boundary value problems for some values of the parameters characterizing these processes, pressure distribution graphs were obtained. In all cases, ultrasonic beams are weaken monotonically with time and completely fade away in a finite time. At each point, the time of complete attenuation of ultrasonic beams is found.

The obtained and studied new submodels are another step towards the creation of a database of physically significant submodels of the three-dimensional generalized model of nonlinear hydroacoustics by Khokhlov-Zabolotskaya-Kuznetsov in a cubic nonlinear medium in the absence of dissipation, which describes the attenuation of ultrasonic beams after the formation of shock fronts.

Submodels of the attenuation of ultrasonic beams in 3-d cubic nonlinear medium in the absence of dissipation

Yu. A. Chirkunov and M. Yu. Chirkunov

Department of Information and Economic Systems, Reshetnev Siberian State University of Science and Technology, Krasnoyarsk, Russia; chr101@mail.ru, mihail0807@mail.ru

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General solution of the Maxwell equations to the stagnation point problem with cylindrical symmetry

C. Chittam[1], *S. V. Meleshko*[2]

School of Mathematical Sciences and geoinformatics, Suranaree University of Technology, Nakhon Ratchasima, Thailand; [1]nursejaypark@gmail.com, [2]sergey@math.sut.ac.th

The behavior of a fluid near the critical point is considered using Maxwell's equations. The model under study is the Johnson-Sigalman model. Although the study of stagnation point flow problems has been studied frequently, general exact analytical solutions for the cylindrical case, which are more suitable for certain experiments, have not been explored. In this study, we obtained the general solution of Maxwell's equations for the cylindrical case of the critical point problem. The presentation is focused to the upper convection derivative.

Symmetries and conservation laws for differential equations, difference equations and second-order delay ODEs

V. A. Dorodnitsyn

Keldysh Institute of Applied Mathematics, Russian Academy of Science, Miusskaya Pl.4, Moscow, 125047, Russia; dorod2007@gmail.com

We discuss the Noether theorem for differential and difference equations, Nother's operator identity and conservation laws. We also consider the Hamilton identity, which provides with simple and quick link to first integrals for ODEs and for difference ODEs. Based on the Lagrangian identity the method of adjoint equations gives the possibility to find conservation laws for equations without Lagrangian and Hamiltonian. We show the connection of this method with so called direct method. We discuss also the Lagrangian formalism for variational delay ordinary differential equations. The Noether operator identity is used to formulate the Noether-type theorems, which allow to find first integrals for delay ODEs.

The presentation is based on the joint works with R.Kozlov, P.Winternitz, S.Meleshko and E.Kaptsov.

Multidimensional Riemannian metrics and their application for the integration of a system of Navier-Stokes equations

V. Dryuma

IMI of the State University of Moldova, Chisinau, Moldova; valdryum@gmail.com

The report will consider examples of metrics of multidimensional Riemannian spaces associated with the Navier-Stokes (NS) system of equations. In the case of a 14D metric with Riemann curvature R_{ijkl} and its Ricci curvature tensor R_{ik}, depending on the variety of the Lie group E^8, it has four components that vanish, provided that the functions of pressure and velocity of the liquid satisfy the equations NS. The 8D-space metric is a composition of two 4D dual metrics with scalar Cartan invariants and is used to study the properties of flows with the Eulerian and Lagrangian approaches in describing the properties of flows of a viscous incompressible fluid. In the case of 6D-space, the Ricci tensor of the metric has nonzero components depending on the velocity functions U, V, W and is used to study the properties of its Ricci flows. The behavior of geodesic lines of metrics depends on the invariants of Cartan of the metrics and can be both regular and chaotic.

Some properties of completely integrable Hamiltonian systems

A. Ya. Narmanov[1], *Sh. R. Ergashova*[2]

Department of Mathematics, National University of Uzbekistan, Tashkent, Uzbekistan;
[1]narmanov@yandex.com, [2]shohida.ergashova@mail.ru

Definition-1 Let M^m (where $m = 2n$) be a Poisson manifold and $sgradH$ Hamiltonian vector field with a smooth Hamiltonian function H.

Hamiltonian system $sgradH$ is called *completely integrable in the sense of Liouville or completely integrable*, if exists set of smooth functions $f_1, ..., f_n$ as:

1) $f_1, ..., f_n$ are first integrals of $sgradH$ Hamiltonian vector field,

2) they are functionally independent on M, that is, almost everywhere on M their gradients are linearly independent,

3) $\{f_i, f_j\} = 0$ for any i and j,

4) the vector fields $sgradf_i$ are complete, that is natural parameter on their integral trajectories is defined on the whole number line [3].

Definition-2 Partition of the manifold M^m into connected components of joint level surfaces of the integrals $f_1, ..., f_n$ is called *The Liouville foliation* corresponding to the completely integrated system.

Let $sgradH$ be a completely integrable Hamiltonian vector field and with Hamiltonian function $H: \mathbb{R}^4 \to \mathbb{R}$ on the four dimensional Euclidean space with the Cartesian coordinates (p_1, p_2, q_1, q_2) with equation:

$$H = H(p_1, p_2, q_1, q_2) \tag{1}$$

The Hamiltonian vector field corresponding to H [1] is

$$sgradH = -\frac{\partial H}{\partial q^1}\cdot\frac{\partial}{\partial p^1} - \frac{\partial H}{\partial q^2}\cdot\frac{\partial}{\partial p^2} + \frac{\partial H}{\partial p^1}\cdot\frac{\partial}{\partial q^1} + \frac{\partial H}{\partial p^2}\cdot\frac{\partial}{\partial q^2}, \tag{2}$$

where Hamiltonian system has following form

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \quad i = 1, 2. \tag{3}$$

We assume that Hamiltonian system is completely integrable and following functions

$$F^1 = F^1(p_1, q_1), \quad F^2 = F^2(p_2, q_2) \tag{4}$$

are first integrals of Hamiltonian system (3).

Level surfaces of these first integrals generates Liouville foliation. If the dimension of the leaf L is maximal, it is called regular, otherwise L is called singular.

Theorem. Regular leaves of Liouville foliation generated by Hamiltonian system (3) are two dimensional surfaces with zero Gauss curvature and zero Gauss torsion.

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On the numerical and analytical solution of the Cauchy problem for ideal plasticity

D. O. Evtikhov

Reshetnev Siberian State University of Science and Technology, Russia;
devtikhov@yandex.ru

The paper considers the construction of characteristics of the Cauchy problem for a particular planar problem at y=0 using conservation laws The Cauchy problem, one of the main problems of the theory of differential equations, consists in integrating a differential equation satisfying boundary conditions.

We set the Cauchy problem for a plasticity system that determines the stress state of a plastic medium under plane deformation. In the plane of variables x y, the line L is set: y=0,. Smooth functions $\sigma = \sigma_0(x), \theta = \theta_0(x)$ are given on L, the solution of which is continuous together with derivatives up to and including the second order. It is necessary to find a solution in the vicinity of L. Let's set the points S,P on L$S \leq x \leq P$. By releasing the characteristic PR:$\xi = \frac{\sigma}{2k} - \theta$ from point P and the characteristic SR:$\eta = \frac{\sigma}{2k} + \theta$ from some point S. Before their intersection at point R, then the solution of the Cauchy problem is defined in a curved triangle SPR

It is necessary to determine the coordinates of the point R of the intersection of the characteristics. The solution at point R depends only on the data on line L. Knowing the coordinates of the points R, it is possible to construct sliding lines formed by these points.

It is worth noting that the Cauchy problem has a solution if: the characteristics of one family do not intersect, the line L is not a characteristic of the system of Levy equations and each characteristic of this system can intersect it only once.

We checked the calculations under various boundary conditions and found out that not all boundary conditions have a solution to the Cauchy problem, in some cases the characteristics of one family intersect, which makes the values along them different and violates the continuity of the solution.

Thus, the Cauchy problem does not have solutions for all boundary conditions, but it is impossible to check these conditions at the present time, since when setting the problem, we do not know the characteristics and cannot guarantee its correct formulation until the problem itself is solved. Therefore, it is necessary to develop requirements for setting boundary conditions under which the Cauchy problem would have a solution, this problem will be considered in future works.

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Helicity in dispersive continuum mechanics

S. L. Gavrilyuk

Aix-Marseille University, France; sergey.gavrilyuk@univ-amu.fr

New conservation laws generalizing the helicity integrals are obtained for a class of dispersive models of fluid mechanics. Applications to specific dispersive models are discussed (Euler-van der Waals- Korteweg fluids, Serre-Green-Naghdi equations, equations of bubbly fluids, ...).

This is a joint work with Henri Gouin.

Hidden Attractors in Symmetric Gene Networks Models

A. V. Glubokikh[1], *V. P. Golubyatnikov*[2]

Department of Mathematics, Novosibirsk State University, Novosibirsk, Russia;
[1]a.glubokikh@g.nsu.ru, [2]vladimir.golubyatnikov1@fulbrightmail.org

We study 3D nonlinear dynamical system

$$\frac{dx_1}{dt} = L(x_3) - x_1; \quad \frac{dx_2}{dt} = L(x_1) - x_2; \quad \frac{dx_3}{dt} = L(x_2) - x_3; \tag{1}$$

considered as a model of a simple molecular repressillator.

Here, $x_1(t)$, $x_2(t)$, $x_3(t)$ denote concentrations of its components, and the two-steps monotonically decreasing function L is defined as follows: $L(w) = 2a$ for $0 \leq w < a - \varepsilon$, $L(w) = a$ for $a - \varepsilon \leq w < a + \varepsilon$, and $L(w) = 0$ for $a + \varepsilon \leq w$. This function describes negative feedbacks in the gene network.

Similar gene networks were studied in [1]. Note that the cube $Q = [0, 2a] \times [0, 2a] \times [0, 2a]$ is positively invariant domain of the system (1). We decompose Q by six planes $x_j = a-\varepsilon$, $x_j = a+\varepsilon$, $j = 1, 2, 3$, to 27 blocks. The system (1) has a very simple linear formn each of these blocks, and we enumerate them by multi-indices $\{s_1 s_2 s_3\}$ so that $s_j = 0$ if $0 \leq x_j < a - \varepsilon$ in this block; $s_j = 1$ if $a - \varepsilon \leq x_j < a + \varepsilon$; and $s_j = 2$ if $a + \varepsilon \leq x_j$. The system (1) is symmetric with respect to the cyclic permutations of the variables

$$x_1 \to x_2 \to x_3 \to x_1.$$

Theorem 1. *If $6\varepsilon < a$ then the system (1) has a stable equilibrium point $S_0 = (a, a, a)$; at the same time, the cube Q contains a piecewise linear cycle composed by 12 segments, all of them are located sufficiently far from the block* $\{111\}$.

So, this point $S_0 \in \{111\}$ is the hidden attractor of the system (1).

Previously, similar phenomena, including non-uniqueness of cycles of similar dynamical systems, were known only in higher-dimensional cases, see [2, 3].

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Application of a Fractional Derivative to Simulate the Evolution of the Interfacial Surface in a Bubbling Bubble under the Influence of Ultrasonic Vibrations in a Liquid

R. N. Golykh

Chair of Methods and Means of Measurement and Automation, Biysk Technological Institute (branch) of Polzunov Altai State Technical University, Biysk city, Altai region, Russian Federation; romangl90@gmail.com

Mass transfer processes of a gas dissolved in a liquid are slow diffusion processes in comparison the mass transfer of a gas dissolved in a gas or a liquid dissolved in another liquid. This is due to the low diffusion coefficient of gas in liquid, which is 5000...10000 times less than the diffusion coefficient of gas in gas [1].

Due to this physical limitation, it is necessary to increase the overall mass transfer rate by increasing the interfacial surface. In turn, an increase in the integral gas flow, achieved with an increased surface, will increase the rate of physical and chemical processes (gas dissolution - absorption at an increased partial pressure of the target component in the gas phase, degassing at a reduced partial pressure of the target component in the gas phase, chemical reactions of gas with liquid) in the surface layer near the liquid-gas interface [2]. One of the most effective ways to increase the interfacial surface is to create ultrasonic cavitation in the continuous liquid phase, which makes it possible to excite surface waves on stable gas bubbles formed as part of the bubbling process [3].

The diffusion-thermodynamic model of the evolution of the size of the spherical gas bubble in the spherical approximation was proposed. The closed system of equations for the evolution of the bubble size, the distribution of concentrations of gas components in it, and the temperature distribution is constructed. The method was proposed for representing solutions in the form of integrals of elementary functions of the following kind

$$e^{-k^2 t - \frac{kr}{DR}} f_k\left(t, \frac{r}{R}\right); \tag{1}$$

where k is scale of parameter's (concentration or temperature) gradient, m^{-1}; R is bubble radius, m; t is time, s; D is diffusion or temperature conductivity coefficient, $\frac{m^2}{s}$; f_k is function taking into account bubble radius changing.

Based on the proposed representation and using the properties of uniform convergence of integrals, approximate equations were obtained that describe the evolution of the concentrations of gas components and temperature near the boundary of the bubble along with the radius of the latter using the apparatus of fractional time derivatives [4]. The numerical solution of the proposed equations made it possible to find the ranges of ultrasonic exposure modes that ensure the stable existence of a bubble bubble without collapse, depending on its size and the initial content of gas components. Next, having found the conditions for the stable existence of a bubble bubble, an analysis of the formation of capillary waves was carried out. An equation has been constructed for the formation of a wave of a fixed length on the surface of a bubble based on the exact solution of the Laplace equation for the velocity potential of the surrounding fluid with the boundary conditions of a "capillary jump". Based on the constructed partial differential equation, an equation is derived for the formation of waves of an arbitrary profile, but of small amplitude, throughout the entire shell of the bubble using the apparatus of fractional derivatives with respect to time. Using multiple differentiation by time, an equation with integer derivatives was constructed. The new equation contains the 4th order time derivative and the triple Laplace operator in the tangent coordinate system. The tangent coordinate system is the system in which the x,y plane coincides with the tangent plane at a given point, and the z axis is perpendicular to the tangent plane. Next, using an equation with integer derivatives, the method was proposed for calculating the eigen frequencies of oscillations of a spherical bubble, which provide capillary wavelengths that are small compared to the radius of the bubble. Finally, the influence of the curvature of a spherical surface on the distortion of capillary waves is analyzed.

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Theoretical and Experimental Study of the Evaporative Convection Induced by Gas Pumping

O. N. Goncharova[1,2], *V. B. Bekezhanova*[2], *Yu. V. Lyulin*[2,3]

[1]*Department of Differential Equations, Altai State University, Barnaul, Russia*; gon@math.asu.ru

[2]*Department of Differential Equations of Mechanics, Institute of Computational Modelling SB RAS, Krasnoyarsk, Russia*; vbek@icm.krasn.ru

[3]*National Research University, "Moscow Power Engineering Institute", Moscow, Russia*; LiulinYV@mpei.ru

The study of the convective processes with evaporation in the two-layer systems is one of the actual research directions in the field of thermophysics and hydrodynamics. The developed experimental methods, including the experimental rig, optical techniques, PIV method and infrared photography, make it possible to measure the evaporative mass flow rates, interface temperature and surface temperature gradients, shear and thermocapillary stresses. These flow characteristics were investigated for the ethanol (HFE-7100) – air (nitrogen) working systems with a layer thickness in the range of [1–8] (mm). New experimental data were obtained for the gas flow velocity varied from 0.001389 to 1.389 (m/s). When choosing the input data for theoretical studies, the working parameters admitted by the experimental setup and the available experimental data determining the thermal conditions and based on thermograms are taken into account.

The flows in the liquid and gas-vapor systems are studied analytically and numerically on the basis of the Oberbeck – Boussinesq approximation of the Navier – Stokes equations. The governing equations and boundary relations take into account additionally the thermodiffusion and diffusive thermal conductivity effects occurring in the vapor-gas layer due to the presence of an evaporated component. The Ostroumov – Birich type exact solutions of the convection equations having the group nature are obtained to describe the flows in the infinite horizontal channels in the 3D case. The flows appear in the transversely directed gravity field under the action of the longitudinal temperature gradient. In the 3D case the analytical representations of the unknown functions cannot be constructed only by direct integration. A reduction procedure to a chain of the two-dimensional statements is carried out in order to organize numerically construction of the solution.

Analytical and numerical studies of the two-phase flows with the diffusive-type evaporation or condensation at the interface demonstrate the flow regimes observed in the real physical systems. The influence of different boundary conditions for the temperature and vapor concentration functions on the flow topology and the thermal and concentration fields was studied. Comparison of the calculated values of the evaporation rate and temperature drops in the system obtained on the basis of mathematical model with those measured in the experiments is performed. This enables to identify meaningful formulations of the boundary value problems and to indicate the ranges of variation of the control parameters provided an adequate description of the processes under study. Theoretical results found a good qualitative and quantitative confirmation by experimental data.

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Symmetry transformations of the vortex field statistics in optical turbulence

V. N. Grebenev, S. B. Medvedev

Federal Research Center for Information and Computational Technologies, Novosibirsk, Russia; vngrebenev@gmail.com

We use the concept of gauge transformations in the proof of the invariance of the statistics of zero-vorticity lines in the case of the inverse energy cascade in wave optical turbulence; we study it in the framework of the hydrodynamic approximation (M.D. Bustamante, S.V. Nazarenko [Phys. Rev. E. **92**, 2015]) of the two-dimensional nonlinear Schrödinger equation for the weight velocity field u. The multipoint probability distribution density functions f_n of the vortex field $\Omega = \nabla \times u$ satisfy an infinite chain of Lundgren-Monin-Novikov equations (statistical form of the Euler equations). The equations are considered in the case of the external action in the form of white Gaussian noise and large-scale friction, which makes the probability distribution density function statistically stationary. The main result is that the transformations are local and conformally transform the n-point statistics of zero- vorticity lines or the probability that a random curve $x(l)$ passes through points $x_i \in \mathbb{R}^2$ for $l = l_i$, $i = 1, \ldots, n$, where $\Omega = 0$ is invariant under conformal transformations.

Equivalence group and invariant solutions of the inhomogeneous Boltzmann equations for a binary mixture of gases

Yu. N. Grigoriev[1], *S. V. Meleshko*[2]

[1]*Institute of Computational Technology, Novosibirsk, Russia 630090*; grigor@ict.nsc.ru
[2]*School of Mathematical Sciences and Geoinformatics, Suranaree University of Technology, Nakhon Ratchasima, Thailand*; sergey@math.sut.ac.th

Systems of Boltzmann kinetic equations with integrals of pair elastic collisions that describe the evolution of gas mixtures at the molecular level, extremely complicated mathematically. Currently actually only one class of exact solutions is known, which does not invert the collision integral to identical zero [1]. Accounting for molecular processes in relaxing chemically reacting gases further complicates the kinetic equations in which integrals of double and triple inelastic collisions appear. In order to consider simpler models in the case of weak manifestation of inelastic processes, it is possible replace the corresponding integrals with some source functions. Wherein sources modeling integrals of inelastic processes should obviously include one form or another dependence on a solution. In [2] the transformation of the distribution function (DF) and time, which made it possible to construct in explicit form a generalized Bobylev–Krook–Wu (BKW) solution for the source function in the product form of the volumetric density of particles (molecules) and DF, was found. In this regard, in [3] an extension of the equivalence group has been proposed to take into account the dependence of source functions on functionals (nonlocal operators) from DF. As a result, in a class of generalized BKW solutions was explicitly constructed for sources linear in DF. In this research, this result is generalized to a system of inhomogeneous Boltzmann equations. A system of equations for a binary mixture of gases is considered, but all the calculations are easily transferred to an arbitrary number of equations. Extension of the Lie group, admitted by the system of homogeneous Boltzmann kinetic equations, which is considered as an equivalence group for inhomogeneous equations was constructed. Conditions have been found under which the transformation from the extended group vanishes sources in the transformed equations. A class of sources linear in DF is identified, for which generalized Bobylev–Krook–Wu solutions are obtained explicitly, in particular, modeling the kinetics of dissociation and recombination processes.

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Symmetries of (1+2)-dimensional Jaulent-Miodek Hierarchy

A. K. Halder[1], *R.Seshadri*[2], *A. Paliathanasis*[3], *P. G. L. Leach*[4]

[1]*School of Technology, Woxsen University, Hyderabad, Telangana 502345, India*; amlanhalder1@gmail.com; amlankanti.halder@woxsen.edu.in

[2]*Department of Mathematics, Pondicherry University, Puducherry - 605014, India*; seshadrirajeswari@gmail.com

[3] *Institute of Systems Science, Durban University of Technology, Durban 4000, South Africa and Departamento de Matemáticas, Universidad Católica del Norte, Avda. Angamos 0610, Casilla 1280 Antofagasta, Chile*; anpaliat@phys.uoa.gr

[4] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa and Institute for Systems Science, Durban University of Technology, Durban, South Africa*; leachp@ukzn.ac.za

The (1+2)− dimensional Jaulent-Miodek hierarchy represents the Energy-dependent potentials of the nonlinear Schrödinger equation in higher-dimension[1]. The group analysis of the members of the JM hierarchy is conducted to explicitly list out the similarities and dissimilarities with respect to the wave propagation between the members[2, 3]. The point symmetry analysis of the fourth member of the hierarchy is quite similar to that of the first member whereas significant difference is observed between the first three members. The integarbility of the reduced equations will be deduced by the procedure of Singularity analysis.

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Invariant Solutions of Nonlinear Mathematical Modeling of Natural Phenomena

Ranis Ibragimov

Mathematics & Computer Science, De Gruyter, Boston, MA, USA;
Ranis.Ibragimov@degruyter.com

The main objective is to demonstrate the advantages of the invariance method in obtaining new exact analytic solutions expressed in terms of elementary functions for various physical phenomena. As one particular application of the invariance method will be the mathematical modeling of oceanic and at mospheric whirlpools causing weather instabilities and, possibly, linked with climate change. As another particular example, it will be demonstrated that the invariance method allows to obtain the exact solutions of fully nonlinear Navier-Stokes equations within a thin rotating atmospheric shell that serves as a simple mathematical description of an atmospheric circulation caused by the temperature difference between the equator and the poles with included equatorial flows modeling heat waves, known as Kelvin Waves. Special attention will be given to analyzing and visualizing the conserved densities associated with obtained exact solutions. As another modeling scenario, the exact solution of the shallow water equations simulating equatorial atmospheric waves of planetary scales will be analyzed and visualized.

A Second Order Ordinary Differential Equation for the Riemann Zeta Function

Asghar Qadir[1], *Aamina Jamshaid*[2]

[1]*Pakistan Academy of Sciences, 3 Constitution Avenue, Islamabad, Pakistan*; asgharqadir46@gmail.com

[2]*College of Electrical & Mechanical Engineering, National University of Sciences & Technology, Peshawar Road, Rawalpindi, Pakistan*; aaminajamshaid@gmail.com

One of us (AQ [1]) had noted a connection between Singularity Analysis and a Distributional Representation for Special Functions, in that both led to remarkable results due to singularities in the complex plane. It was further observed that the connection allowed the possibility of obtaining a second order Ordinary Differential Equation for the Riemann Zeta function, which did not arise from one (as most special functions do.) Here we present the differential equation for the real part of the independent variable.

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Theory of asymptotic solitons for non-integrable equations

A. M. Kamchatnov[1,2]

[1]*Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, Russia*;
kamch@isan.troitsk.ru
[2]*Skolkovo Institute of Science and Technology, Skolkovo, Moscow, Russia*;
An.Kamchatnov@skoltech.ru

As is known, the completely integrable soliton equations are associated in the inverse scattering transform method with certain linear spectral problems, so that solitons' characteristics are encoded by the parameters of the discrete spectrum of the corresponding solution. If the number of solitons is large, then we can use the asymptotic WKB method for calculation of the spectrum which provides, consequently, the parameters of solitons at asymptotically large time of evolution. In case of not completely integrable equations, the associated linear spectral problems do not exist and the above approach becomes inapplicable. We suggest another approach based on the Gurevich-Pitaevskii theory of dispersive shock waves which can be applied to both integrable and non-integrable equations. According to this theory, the transition from the initial intensive smooth pulse to the asymptotic state of soliton trains occurs via an intermediate stage of formation and evolution of dispersive shock waves, and the nonlinear oscillations in the shocks transform eventually to the asymptotic solitons. The number of oscillations entering into the shock region per a second is given by the formula

$$\frac{dN}{dt} = \frac{1}{2\pi}\left(k\frac{\partial\omega}{\partial k} - \omega\right),$$

where $\omega = \omega(k)$ is the dispersion relation for linear waves propagating along a uniform background. Integration of this formula over time yields the expression for the Poincaré-Cartan integral invariant

$$N = \frac{1}{2\pi}\oint(k\delta x - \omega\delta t), \qquad \delta x = \frac{\partial\omega}{\partial k}\delta t.$$

We show that this integral is preserved by the dispersionless hydrodynamic flow

$$\frac{\partial r_\pm}{\partial t} + v_\pm\frac{\partial r_\pm}{\partial x} = 0$$

(written here in terms of the Riemann invariants $r_\pm$), if the wave number $k = k(r_+, r_-)$ is only a function of the dispersionless variables and satisfies the equations

$$\frac{\partial k}{\partial r_+} = \frac{\partial\omega/\partial r_+}{v_+ - \partial\omega/\partial k}, \qquad \frac{\partial k}{\partial r_-} = \frac{\partial\omega/\partial r_-}{v_- - \partial\omega/\partial k}.$$

These derivatives commute in case of completely integrable equations and for non-integrable equations we can find asymptotic solutions $k = k(r_+, r_-, q)$ correct for large values of k (q is an integration constant). Due to preservation of the Poincaré-Cartan integral invariant, this solution can serve as an integrand function in the generalized Bohr-Sommerfeld quantization rule

$$\int_{x_1(q_n)}^{x_2(q_n)} k[r_{+,0}(x), r_{-,0}(x), q_n)dx = 2\pi n, \quad n = 1, 2, \ldots, N,$$

where $r_{\pm,0}$ are the initial distributions. As a result, the parameters q_n, $n = 1, 2, \ldots, N$, determine the characteristics (velocities) of the asymptotic solitons [1].

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On Group Foliations, Invariant Solutions, and Conservation Laws of the Geopotential Forecast Equation

E. I. Kaptsov

School of Mathematical Sciences and Geoinformatics, Suranaree University of Technology, Nakhon Ratchasima, Thailand; evgkaptsov@math.sut.ac.th

The geopotential forecast equation is employed to predict the geopotential over areas characterized by rotating air masses situated at intermediate altitudes within the atmosphere. These vortices are often associated with various weather phenomena and can play a significant role in the development of weather systems. This equation is often written as [5, 8]

$$\zeta_t - H_y \zeta_x + H_x(\zeta + f_0 + \beta y)_y = 0,$$

where β and f_0 are constants, $(-H_y, H_x)$ is the two-dimensional velocity potential, $\zeta = H_{xx} + H_{yy}$ represents relative vorticity, $f_0 + \beta y$ is the β-approximation of the Coriolis parameter, and $\zeta + f_0 + \beta y$ is the absolute velocity.

Despite the large number of publications on symmetry analysis of the geopotential forecast equation [5, 8, 9], its group foliations [4] and conservation laws have not previously been considered or were only briefly mentioned. The results presented in [1] aim to address these shortcomings. In [1], group foliations are constructed for the equation, and based on them, invariant solutions are derived, some of which generalize previously known exact solutions. Then, all possible second-order conservation laws of the geopotential forecast equation are obtained through direct calculations, and a number of higher-order conservation laws are derived using the known symmetries of the equation.

The work also adds to the list of known applications of the group foliation approach (such as [2, 3, 4, 6, 7]). This list is currently not extensive enough, and new examples of the use of group foliations should be of interest to specialists in the group analysis of differential equations.

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Contact Mappings of Jet Spaces

O. V. Kaptsov

Institute of Computational Modelling SB RAS, Krasnoyarsk, Russia;
kaptsov@icm.krasn.ru

As is well known, contact transformations are used to solve problems of classical mechanics and equations of mathematical physics [1]. The most known examples of such transformations are the Legendre and Ampere transformations. The theory of contact transformations was developed by S. Lie. At present, there are numerous sources devoted to these issues [2]. The contact transformations are diffeomorphisms of the jet space that preserve the contact structure. To integrate differential equations, it is useful to find contact transformations that leave these equations invariant.

However, not only contact transformations are applied to integrate differential equations. Leonhard Euler started using differential substitutions, which are not diffeomorphisms, to integrate linear partial differential equations [3]. Now these substitutions are called the Euler- Darboux transformation or simply the Darboux transformation.

In this report, we consider analytic mappings of jet spaces that preserve the modulus of canonical differential forms and call these mappings contact. We prove a lifting lemma that shows how to construct a contact mapping. For applications to differential equations, the mappings are required to transform solutions of the equations into solutions of other equations or act on solutions of given equations. Examples of second-order partial differential equations connected by contact mappings are given.

We also study contact mappings depending on a parameter. It is easier to look for such mappings in the form of series in powers of the parameter. As an example, we consider the Burgers equation. Parametric contact mappings are found that act on solutions of this equation. These mappings have no inversional maps [4].

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On Classical Leray Problems for Steady–State Navier–Stokes system

M. V. Korobkov[1,2]

[1]*School of Mathematical Sciences, Fudan University, Shanghai, China*;
korob@math.nsc.ru
[2]*Sobolev Institute of Mathematics, Novosibirsk, Russia*;

In recent years, using the geometric and real analysis methods, essential progress has been achieved in some classical Leray's problems on stationary motions of viscous incompressible fluid: the existence of solutions to a boundary value problem in a bounded plane and three-dimensional axisymmetric domains under the necessary and sufficient condition of zero total flux; the uniqueness of the solutions to the plane flow around an obstacle problem in the class of all D-solutions, the nontriviality of the Leray solutions (obtained by the "invading domains" method) and their convergence to a given limit at low Reynolds numbers; and, more generally, the existence and properties of D-solutions to the boundary value problem in exterior domains in the plane and three-dimensional axisymmetric case, etc. A review of these advances and methods will be the focus of the talk. Most of the reviewed results were obtained in our joint articles with Konstantin Pileckas, Remigio Russo, Xiao Ren, and Julien Guillod, see, e.g., the recent survey paper J. Math. Fluid Mech. **25** (55) (2023), http://dx.doi.org/10.1007/s00021-023-00792-w

Poroelastic Problem of a Non-Penetrating Crack with Cohesive Contact for Fluid-Driven Fracture

Victor A. Kovtunenko[1,2]

[1]*Department of Mathematics and Scientific Computing, Karl-Franzens University of Graz, NAWI Graz, Austria*; victor.kovtunenko@uni-graz.at

[2]*Lavrentyev Institute of Hydrodynamics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russia*;

We introduce a new class of unilaterally constrained problems for fully coupled poroelastic models stemming from hydraulic fracturing and study its well-posedness. The poroelastic medium contains a fluid-driven crack, which is subjected to non-penetrating conditions and cohesion forces between the crack faces [4]. Compared to the classical model of a hydraulically open fracture, non-penetration allows compression at which the fracture can be mechanically closed [10, 11]. Solvability of the governing elliptic-parabolic variational inequality under the unilateral constraint with a small cohesion is established using the incremental approximation based on Rothe's semi-discretization in time [6].

For the poroelastic system with cohesionless non-penetrating crack, the incremental model is expressed by a saddle-point problem with respect to the unknown solid phase displacement, pore pressure, and contact force [3, 7, 9]. Applying the Lagrange multiplier approach and Delfour–Zolesio theorem, formula of the shape gradient under crack perturbation is derived [8]. Ii is useful for finding numerical solution by minimization schemes of gradient type [1, 5]. In the plane isotropic setting, a Fourier series solution is obtained in the sector of angle 2π with respect to distance to the crack-tip [2]. A square-root singularity takes place, and no logarithmic terms occur in the asymptotic expansion. Integral formulas calculating stress intensity factors are rigorously calculated.

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Bifurcations of phase portraits, exact solutions and conservation laws of the generalized Gerdjikov - Ivanov model

N. A. Kudryashov, S. F. Lavrova, D. R. Nifontov

National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation; nakudryashov@mephi.ru

This work explores the generalized Gerdjikov-Ivanov equation describing propagation of pulses in optical fiber in the form,

$$i\,q_t + a\,q_{xx} + b\,|q|^4\,q + i\,c\,q^2\,q_x^* = i\,\left[\alpha\,q_x + \lambda\,(|q|^{2m}\,q)_x + \mu\,(|q|^{2m})_x\,q\right], \tag{1}$$

where $q(x,t)$ is a complex-valued function, which describes the wave profile, a, b, c, α, λ and μ are parameters of the mathematical model, where a is responsible for group velocity dispersion, b is the coefficient of quintic nonlinearity, α is the coefficient of intermodal dispersion, c and μ are coefficients of nonlinear dispersion and λ is the coefficient of the self-steepening term for short pulses.

Equation (1) is a well-known nonlinear partial differential equation for description of optical solitons in fiber, especially in photonic crystal fibers. This equation does not pass the Painlevé test and the Cauchy problem for Equation (1) cannot be solved by the inverse scattering transform in the general case. In this regard, analytical solutions for the generalized Gerdjikov-Ivanov equation are found using traveling wave variables. However, at $\alpha = \lambda = \mu = 0$ Equation (1) is an integrable equation, which has been shown in a paper [1].

Equation (1) has been considered at $m = 1$ in a number of articles. In the paper [2] the authors have generated new optical soliton solutions to the perturbed Gerdjikov Ivanov equation which have been detected by means of the extended direct algebraic method. The perturbed Gerdjikov-Ivanov equation which describes the dynamics of the soliton in an optical fiber has been investigated in the paper [3].

In this work we obtain the nonlinear ordinary differential equation corresponding to Equation (1), periodic and solitary wave solution of this ordinary differential equation at m = 1 and m = 2 and in the case of arbitrary value m exact solutions in the form of optical solitons. Phase portraits of an ordinary differential equation corresponding to the partial differential equation under consideration are constructed and have presented the classification of the phase portraits corresponding to this equation.

Three conservation laws for the generalized equation corresponding to power conservation, moment and energy are found by the method of direct transformations.

Conservative densities corresponding to optical solitons of the generalized Gerdjikov - Ivanov equation, have been received. The conservative quantities obtained have not been presented before in the literature, to the best of our knowledge. These theoretical results obtained can be useful for practical applications due to them being helpful in testing, whether numerical schemes for partial differential equations are conservative.

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Riemann problem for rate-type materials with non-constant initial conditions

R. Radha[1], *V. D. Sharma*[2], *Akshay Kumar*[3]

[1]*School of Mathematics and Statistics, University of Hyderabad, Hyderabad 500046, India*; repakar@yahoo.com
[2]*Indian Institute of Technology Gandhinagar, Palaj, Gujarat 382355, India.*; vsharma@iitgn.ac.in
[3]*School of Mathematics and Statistics, University of Hyderabad, Hyderabad 500046, India*; a.kumar119119@gmail.com

In this talk, the compatible theory of differential invariants will be discussed and a class of exact solutions is obtained for non-homogeneous quasi-linear hyperbolic system of partial differential equations (PDEs) describing rate-type materials; these solutions exhibit genuine non-linearity that leads to the formation of discontinuities such as shocks and rarefaction waves. For certain non-constant and smooth initial data, the solution to the Riemann problem is presented providing a complete characterization of the solutions.

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Dynamical Behaviour of Coupled Burgers' Equations Arising in Fluid

Raj Kumar, Kripa Shankar Pandey and Avneesh Kumar[1]

Department of Mathematics, Faculty of Engineering and Technology
Veer Bahadur Singh Purvanchal University, Jaunpur–222003, India;
[1]avnee2864@gmail.com

Nowadays, coupled partial differential equations have been employed in various fields of engineering and applied sciences. Coupled Burgers' equations are coupled PDEs, and describe the approximation theory of flow through a shock wave traveling in a viscous fluid. In this research article, the Lie symmetry analysis of the system was determined, and symmetry reductions were obtained from the coupled Burgers' equations. The authors generated travelling wave solutions with the aid of translation symmetries in time and space. The results obtained by using the "Lie symmetry analysis" method are compared with others. In order for better understanding, the analytical solutions are graphically depicted.

Solutions of Magnetohydrodynamics Equation Through Symmetries

R. Sinuvasan[1], *A. K. Halder*[2], *R.Seshadri*[3], *A. Paliathanasis*[4], *P. G. L. Leach*[5]

[1] *Department of Mathematics,VIT-AP University, Amaravati-522237, Andhra Pradesh, India*; rsinuvasan@gmail.com

[2] *School of Technology, Woxsen University, Hyderabad, Telangana 502345, India*; amlanhalder1@gmail.com; amlankanti.halder@woxsen.edu.in

[3] *Department of Mathematics, Pondicherry University, Puducherry - 605014, India*; seshadrirajeswari@gmail.com

[4] *Institute of Systems Science, Durban University of Technology, Durban 4000, South Africa and Departamento de Matemáticas, Universidad Católica del Norte, Avda. Angamos 0610, Casilla 1280 Antofagasta, Chile*; anpaliat@phys.uoa.gr

[5] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa and Institute for Systems Science, Durban University of Technology, Durban, South Africa*; leachp@ukzn.ac.za

The Magnetohydrodynamics (1+1)–dimension equation with force and force-free term is analysed with respect to its point symmetries[1, 2]. Interestingly, it reduces to an Abel's Equation of the second kind and under certain conditions to equations specified in Gambier's family[3]. The symmetry analysis for the force-free term leads to an Euler's Equation and to a system of reduced second-order odes for which the singularity analysis is performed to determine their integrability.

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On the Instability for One Subclass of Three-dimensional Dynamic Equilibrium States of the Electron Vlasov-Poisson Gas

Yu. G. Gubarev[1,2], *Y. Liu*[2]

[1]*Laboratory for Explosion Physics, Lavrentyev Institute for Hydrodynamics, Novosibirsk, Russian Federation*; gubarev@hydro.nsc.ru

[2]*Department for Differential Equations, Novosibirsk National Research State University, Novosibirsk, Russian Federation*; 243428357@qq.com

In this report, we consider the spatial motions of a boundless collisionless electron Vlasov-Poisson gas in three-dimensional (3D) Cartesian coordinate system:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + \frac{\partial \varphi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0,\ \frac{\partial^2 \varphi}{\partial x_i^2} = 4\pi\Big(\int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} - n_e\Big). \tag{1}$$

Here, $f \geq 0$ denotes the distribution function of electrons (for reasons of convenience, their charges and masses are assumed to be equal to unity); t is time; $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ denote coordinates and velocities of electrons; $\varphi(\mathbf{x}, t)$ is the potential of self-consistent electric field; $d\boldsymbol{v} \equiv dv_1 dv_2 dv_3$; $4\pi n_e \equiv \text{const} > 0$ is the electrons density in some spatial static state of global thermodynamic equilibrium. We suppose that the distribution function f approaches zero asymptotically as $|\mathbf{v}| \to \infty$ and/or $|\mathbf{x}| \to \infty$, and this function along with the potential φ are periodic in argument $\mathbf{x}$ or approach zero asymptotically as $|\mathbf{x}| \to \infty$ too.

It is assumed that mixed problem (1) has the following exact stationary solutions:

$$f = f^0(\mathbf{v}) \geq 0,\ \varphi = \varphi^0 \equiv \text{const};\ \int_{\mathbb{R}^3} f^0(\mathbf{v}) d\mathbf{v} = n_e. \tag{2}$$

The aim of this report is to prove an absolute linear instability for 3D states (2) of dynamic (local thermodynamic) equilibrium of the boundless collisionless electron Vlasov-Poisson gas with respect to small spatial perturbations $f'(\mathbf{x}, \mathbf{v}, t)$ and $\varphi'(\mathbf{x}, t)$:

$$\begin{aligned} &\frac{\partial f'}{\partial t} + v_i \frac{\partial f'}{\partial x_i} + \frac{\partial \varphi'}{\partial x_i}\frac{\partial f^0}{\partial v_i} = 0, \\ &\frac{\partial^2 \varphi'}{\partial x_i^2} = 4\pi \int_{\mathbb{R}^3} f'(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}. \end{aligned} \tag{3}$$

To achieve this aim, a transition from kinetic equations (1) which describe the 3D motions of electron gas under study to an infinite system of relations similar to the equations of isentropic flows of a compressible fluid medium in the "vortex shallow water" and the Boussinesq's approximations was carried out. In the course of instability proof, the well-known sufficient Newcomb-Gardner-Rosenbluth condition for stability of spatial states (2) of dynamic (local thermodynamic) equilibrium in relation to one incomplete unclosed partial class of small 3D perturbations was conversed. Also, some linear ordinary differential second-order inequality with constant coefficients was obtained for the Lyapunov functional. The a priori exponential lower estimate for growth of small spatial perturbations (3) follows from this inequality when the sufficient conditions found in this report for linear practical instability of the considered dynamic (local thermodynamic) equilibrium states are satisfied. Since the obtained estimate was deduced without any additional restrictions on the dynamic equilibrium states under study, then, thereby, the absolute linear instability of 3D states (2) of dynamic (local thermodynamic) equilibrium of the boundless collisionless electron Vlasov-Poisson gas with respect to small spatial perturbations (3) was proved. At last, the analytical examples of stationary solutions (2) and growing perturbations (3) were constructed.

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On the Instability for One Partial Class of Three-dimensional Dynamic Equilibrium States of the Hydrogen Vlasov-Poisson Plasma

Yu. G. Gubarev[1,2], *J. Luo*[2]

[1]*Laboratory for Explosion Physics, Lavrentyev Institute for Hydrodynamics, Novosibirsk, Russian Federation*; gubarev@hydro.nsc.ru

[2]*Department for Differential Equations, Novosibirsk National Research State University, Novosibirsk, Russian Federation*; 644603544@qq.com

In order to prove an absolute instability for one subclass of three-dimensional (3D) dynamic equilibria of hydrogen Vlasov-Poisson plasma in relation to small spatial perturbations, we rely on the fact that in the electrostatic approximation, when there is no the magnetic field, and the electric field is self-consistent, the plasma dynamics is described by the Vlasov-Poisson equations [1]. These equations characterize a collisionless motion of electrons and their interaction with each other through the Coulomb repulsive forces on the background of a uniform distribution of ions inside the physical continuum.

The Vlasov-Poisson mathematical model for a boundless electrically neutral hydrogen plasma in the electrostatic approximation without collisions in a 3D Cartesian coordinate system can be written in the following index form:

$$\begin{cases} \dfrac{\partial f}{\partial t} + v_i \dfrac{\partial f}{\partial x_i} - \dfrac{\partial \varphi}{\partial x_i}\dfrac{\partial f}{\partial v_i} = 0, \\ \dfrac{\partial^2 \varphi}{\partial x_i^2} = 4\pi(1 - \displaystyle\int_{\mathcal{R}^3} f(x_i, v_i, t)\, dv_1 v_2 v_3), \\ f = f(x_i, v_i, t) \geq 0,\ f(x_i, v_i, 0) = f_0(x_i, v_i). \end{cases} \tag{1}$$

Here, f – electron distribution function; x_i, v_i $(i = 1, 2, 3)$ – electron coordinates and velocities; t – time; φ – the potential of self-consistent electric field; f_0 – initial data.

Mathematical model (1) of the hydrogen Vlasov-Poisson plasma has the following exact stationary solutions:

$$f = f^0(v_i),\ \varphi = \varphi^0 \equiv \text{const};\ \int_{\mathcal{R}^3} f^0(v_i) dv_1 dv_2 dv_3 = 1. \tag{2}$$

In the process of proving instability, we use a hydrodynamic substitution of independent variables to transform the Vlasov-Poisson equations (1) into an infinite system of 3D equations which are similar to the equations for isentropic flows of compressible fluid medium in the "vortex shallow water" and the Boussinesq approximations [2]. After that, these new defining equations are linearized in the vicinity of their exact stationary solutions. The direct Lyapunov method [2] is considered to construct a priori exponential estimate from below for one partial class of small spatial perturbations of exact stationary solutions to new defining equations, which grow over time and are described by the field of Lagrangian displacements [2]. As a result, the Newcomb-Gardner-Rosenbluth sufficient condition [3] for linear stability of exact stationary solutions (2) is reversed, its formal character is revealed, and, thus, absolute instability for these solutions is proved. Also, the sufficient conditions for linear practical instability of exact stationary solutions (2) to mathematical model (1) are found, and their constructive nature is discovered. At last, the analytical examples of solutions (2) and growing perturbations are constructed. So, the results obtained can be applied to the development of devices designed to perform the controlled thermonuclear fusion.

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Wave propagation over non-reflective profiles, reaching a constant at infinity

I. E. Melnikov[1,2], *E. N. Pelinovsky*[1,2]

[1]*National Research University Higher School of Economics, Nizhniy Novgorod, Russia*;
[2]*Gaponov-Grekhov Institute of Applied Physics, Nizhniy Novgorod, Russia*;
melnikovioann@gmail.com, pelinovsky@appl.sci-nnov.ru

Wave propagation without reflection is of great importance for applications because it allows energy to be transmitted over long distances. The paper discusses a way to reduce the equations of the linear theory of shallow water to a modified Euler-Poisson-Darboux equation with a variable coefficient in the form of an inverse hyperbolic sine, the solution of which is represented as a composition of traveling waves. Due to this, 2 counting series of non-reflective bottom profiles were obtained. The first series is bounded everywhere, and infinitely smoothly connects the cut with the constant at infinity. The second series, on the contrary, makes the transition from depth to constant. The analysis of the dynamics of the obtained wave fields will be discussed in this report.

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Sedeonic Equations for Fields with Non-Zero Mass of Quantum

V. L. Mironov[1], *S. V. Mironov*

Institute for Physics of Microstructures RAS, Nizhny Novgorod, Russia;
[1]mironov@ipmras.ru

We use the non-commutative (associative) algebra of sixteen-component space-time sedeons [1] to describe fields with non-zero mass of quantum. This algebra takes into account the symmetry of physical quantities with respect to spatial and temporal inversion and allows formulating equations in compact and highly symmetrical form.

The description is based on wave equations using space-time operators of the following form $\widehat{\nabla} = i\mathbf{e}_1\partial - \mathbf{e}_2\nabla - i\mathbf{e}_3 m$, where m is the normalized mass of the quantum. In particular, the second-order wave equation for the sedeonic field potential $\tilde{\mathbf{W}}$ is written in compact form as $\widehat{\nabla}\widehat{\nabla}\tilde{\mathbf{W}} = \tilde{\mathbf{J}}$. This sedeonic equation can be reformulated in equivalent form as the system of Maxwell-like equations

$$\begin{aligned}
&\partial g_1 + \left(\vec{\nabla}\cdot\vec{G}_1\right) - m g_4 = \rho_1\ ,\\
&\partial g_2 + \left(\vec{\nabla}\cdot\vec{G}_2\right) + m g_3 = \rho_2,\\
&\partial g_3 + \left(\vec{\nabla}\cdot\vec{G}_3\right) - m g_2 = \rho_3,\\
&\partial g_4 + \left(\vec{\nabla}\cdot\vec{G}_4\right) + m g_1 \ = \rho_4,\\
&\partial\vec{G}_1 + \vec{\nabla} g_1 - \left[\vec{\nabla}\times\vec{G}_2\right] + m\vec{G}_4 = -\vec{j}_1,\\
&\partial\vec{G}_2 + \vec{\nabla} g_2 + \left[\vec{\nabla}\times\vec{G}_1\right] - m\vec{G}_3 = -\vec{j}_2,\\
&\partial\vec{G}_3 + \vec{\nabla} g_3 + \left[\vec{\nabla}\times\vec{G}_4\right] + m\vec{G}_2 = -\vec{j}_3,\\
&\partial\vec{G}_4 + \vec{\nabla} g_4 - \left[\vec{\nabla}\times\vec{G}_3\right] - m\vec{G}_1 = -\vec{j}_4,
\end{aligned}$$

where g_k and $\vec{G}_k$ are scalar and vector field strengths and ρ_k and $\vec{j}_k$ are sources.

It allows us to formulate the relations for the field energy similar to Poynting's theorem in electrodynamics and apply the classical description of the interaction of particles - the sources of this field. The second-order wave equations can be used to describe baryon fields. As an example, the interaction of two point baryons (participating in the strong interaction) is considered in terms of the overlap of their scalar and vector fields [2].

On the other hand, a first-order wave equation $\widehat{\nabla}\tilde{\mathbf{W}} = \tilde{\mathbf{I}}$ can be used to describe lepton fields. As an example, we consider the interaction of lepton sources caused by the overlap of scalar fields [3].

The gauge (gradient) invariance of the equations for massive fields is also discussed.

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Review of solution procedures of certain ecological models

V. Monishwar Reddy[1], *A. K. Halder*[2], *S. R. Munjam*[3]

School of Technology, Woxsen University, Hyderabad, Telangana 502345, India;
[1]monishwar.reddy_2027@woxsen.edu.in; [2]amlankanti.halder@woxsen.edu.in;
[3]shankar.rao@woxsen.edu.in

In this work, certain ecological models pertaining to various real life problems, such as, population rate of deer in a forest, fish population rate by considering harvesting are studied using the techniques available in the context of dynamical systems. Firstly, the equilibrium of this models are obtained and the behavior of general solution around this equilibrium solution are studied thoroughly. The concept of bifurcation is used to determine the physical significance of solutions obtained by the standard procedures[1].

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Modeling of ”mineral particle-bubble” dynamic in viscous fluid

N. P. Moshkin

Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences, Russia; nikolay.moshkin@gmail.com

The dynamics of a “liquid-bubble-particle” agglomerate in a viscous liquid is a very complicated non-linear, and non-stationary hydrodynamic process. The oscillation of a heavy mineral particle attached to the surface of the gas bubble in the viscous liquid is considered. A bubble that makes surface oscillations and a particle with mass are considered as a unique mechanical system. It is assumed that the main forces determining the interaction of these objects are the inertial force due to surface fluctuations of the gas bubble and the capillary adhesion force. Nonsteady equations describing the dynamics of the mechanical ”liquid-bubble-particle” system in a non-viscous, incompressible fluid are obtained using Lagrangian mechanics. Viscosity influence is accounting by viscous dissipation taking into account. Dynamic behavior of the flotation of the ”bubble-particle” system under various initial disturbances of the bubble surface and particle mass are considered.

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Extensions of Lie algebras and integrability of some equations of fluid dynamics and magnetohydrodynamics

O. I. Morozov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia; oimorozov@gmail.com

We find the twisted extension of the symmetry algebra of the 2D Euler equation in the vorticity form and use it to construct new Lax representation for this equation. Then we consider the transformation Lie-Rinehart algebras generated by finite-dimensional subalgebras of the symmetry algebra and employ them to derive a family of Lax representations for the Euler equation. The family depends on functional parameters and contains a non-removable spectral parameter. Furthermore we exhibit Lax representations for the reduced magnetohydrodynamics equations (or the Kadomtsev-Pogutse equations), the ideal magnetohydrodynamics equations, and the quasi-geostrophic two-layer model equations.

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Integration of Hunter-Saxton-Calogero equation by methods of contact geometry

S. S. Mukhina

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences
Moscow, Russia; ssmukhina@edu.hse.ru

Let's consider the generalized nonlinear second order Hunter–Saxton–Calogero partial differential equation

$$u_{tx} = uu_{xx} + G(u_x), \tag{1}$$

where $u(t,x)$ is an unknown function, t and x are the time and spatial coordinates, respectively. This equation is arised in the theory of control of liquid crystals and in the control of unsteady gas flows [1]. We use the methods developed in [2].

Let $J^1\mathbb{R}^2$ be a 5-dimensional space of 1-jet smooth functions with two independent variables t, x. A contact structure is specified on this space—the Cartan distribution [3]. According to the approach of V.V. Lychagin [4], a Monge–Ampere type equation can be considered as an effective differential 2-form ω on the space $J^1\mathbb{R}^2$. The classical solution of such an equation is a function on whose 1-jet the 2-form ω is canceled, and the multivalued solution is the maximal integral manifold L of the Cartan distribution such that $\omega|_L = 0$.

Equation (1) in the space $J^1\mathbb{R}^2(t, x, u, p_1, p_2)$ corresponds to the effective differential 2-form

$$\omega = -2G(p_2)dt \wedge dx + dt \wedge dp_1 - dx \wedge dp_2 - 2udt \wedge dp_2.$$

To solve the problem of contact linearization of the Monge–Ampere equations A.G. Kushner [5] introduced two differential 2-forms on the space $J^1\mathbb{R}^2$, which he called Laplace forms. This name is justified by the fact that the coefficients of these forms, written for linear hyperbolic equations, exactly coincide with the known Laplace invariants h and k.

In our case, one of the Laplace forms is is equal to zero if and only if $G(p_2) = p_2^2 + 2\alpha_1 p_2 + \alpha_0$, and the other does not vanish. In this case, equation (1) can be linearized using contact transformations.

Applying the Legendre transform

$$\Phi : (t, x, u, p_1, p_2) \to (t, -p_2, -xp_2 + u, p_1, x),$$

we get a linear equation

$$u_{tx} + (x^2 + \alpha_1 x + \alpha_0)u_{xx} + xu_x - u = 0. \tag{2}$$

The equation (2) is solved by the cascade integration method. Knowing the linear solution and applying the inverse contact transformations the original equation was integrated. As a result, the general exact multivalued solution to the Hunter–Saxton–Calogero equation (1) has been founded. The resulting solutions were visualized.

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Analytical Solution of System of Nonlinear Fractional Order Vander Pol Equations

S. R. Munjam[1,2]

[1]*School of Technology, Woxsen University, Hyderabad, Telangana 502345, India*;
shankar.rao@woxsen.edu.in
[2]*Department of Basic Sciences, University of the Peoples, California, USA*;
Shankarrao.Munjam@Uopeople.edu

The study reveals the system of nonlinear fractional order Vander Pol equations (FOVDP) with various conditions are investigated with the method of directly defining inverse mapping (MDDiM). Finding the solutions by using MDDiM is a novel idea and the first time illustrated for the system of nonlinear FOVDP. It is emphasized by residual error (i.e, 10^{-3} to 10^{-17}) and can easily derive deformation terms by spending very low CPU time. Based on the proposed method, the convergence rate, accuracy, and efficiency of the governing equations are demonstrated, which exhibit meaningful structures and advantages in science and engineering. The effects of the variation of all physical parameters are discussed in detail lucidly.

Integration methods based on solvable structures and $\mathcal{C}^\infty$-structures

C. Muriel [1], *A J. Pan-Collantes*[2], *A. Ruiz*[3]

[1]*Department of Mathematics. University of Cádiz. Spain*; concepcion.muriel@uca.es
[2]*Department of Mathematics, University of Cádiz. Spain*; antonio.pan@uca.es
[3]*Department of Mathematics, University of Cádiz. Spain*; adrian.ruiz@uca.es

We present several techniques for solving differential equations based on the existence of solvable structures [2, 7, 1] and $\mathcal{C}^\infty$-structures [4]-[6]. Both concepts are established in a more general framework than that of differential equations, since they provide a systematic integration procedure for involutive distributions of vector fields [8].

Once a solvable structure has been determined for an involutive distribution $\mathcal{Z}$, its elements can be used to construct a sequence of differential 1-forms. Such 1-forms define a succession of completely integrable Pfaffian equations, each defined in a space of dimension one unit smaller than the previous one. These Pfaffian equations can be (locally) solved by quadrature. The successive integration of these equations leads, in the last stage, to the complete integration of the initial distribution $\mathcal{Z}$.

When the elements of a solvable structure are not imposed to be symmetries but $\mathcal{C}^\infty$-symmetries [3], we obtain a larger structure, which has been called $\mathcal{C}^\infty$-structure [4]. Once a $\mathcal{C}^\infty$-structure is known, the integration of the distribution is also achieved by solving at each stage a completely integrable Pfaffian equation. However, the 1-forms defining the Pfaffian equations, unlike in the case of solvable structures, need not be closed. In this scenario, the so-called symmetrizing factors and their relationships with relative integrating factors, recently investigated in [5], are of great use in facilitating the search for primitives at each stage.

Illustrative examples of how both objects (solvable structures and $\mathcal{C}^\infty$-structures) can be found and used to find exact solutions of different problems modeled by differential equations [6] are also presented.

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Differential Invariant Algebras

P. J. Olver

University of Minnesota; olver@umn.edu

A classical theorem of Lie and Tresse states that the algebra of differential invariants of a Lie group or (suitable) Lie pseudo-group action is finitely generated. I will present a fully symbolic, constructive algorithm, based on the equivariant method of moving frames, that reveals the full structure of such non-commutative differential algebras, and, in particular, pinpoints generating sets of differential invariants as well as their differential syzygies. Several applications and outstanding issues will be discussed, including equivalence and symmetry detection in image processing, and some surprising results in classical surface geometries.

First Integrals and Analytical Solutions of Some Tumor Models

H. Öncü[1], *G. Gün Polat*[2], *N. A. Babaei*[3]

[1] *Department of Mathematics, Gebze Technical University, Kocaeli, Turkey*; h.oncu2022@gtu.edu.tr

[2] *Department of Mathematics, Gebze Technical University, Kocaeli, Turkey*; gunpolat@gtu.edu.tr

[3] *Civil Engineering Faculty, Division of Mechanics, Istanbul Technical University, Istanbul, Turkey*; babaei18@itu.edu.tr

Cancer is one of the most deadly diseases among humanity in great part due to the large amount of variables which have to be taken into account in its development and dynamics. Mathematical modelling efficacy and usefulness in providing enough information from which to derive ideas for tumor treatment. This study focuses on the exact analytical solutions of the some tumor models by using artificial Hamiltonian methodology which is a novel algorithm to solve dynamical systems of first-order ordinary differential equations which can be written as a non-standard or partial Hamiltonian system. This method provides an important process to compute the exact solutions of coupled nonlinear systems of ordinary differential equations (ODEs). In this research, our main goal is to analyze two-dimensional nonlinear dynamical systems which are the mathematical models for the evolution of tumor volumes after treatments. Application of the artifical Hamiltonian method to the tumor models enable to determine the first integrals which provides to obtain analytical solutions directly. In addition, graphical representations of the models are presented for the some specifial paremeters.

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A new geometric approach on the linearization of second-order ODEs

A. Paliathanasis

Institute of Systems Science, Durban University of Technology (DUT), Durban 4000, Republic of South Africa.; anpaliat@phys.uoa.gr

We propose a novel method for the linearization of second-order ordinary differential equations. We extend the concept of linearization based on the symmetries to the framework of geometry by using the Eisenhart lift. We establish a new approach for the construction of solutions for differential equations. A demonstration of this approach is the linearization of the Ermakov-Pinney equation and that of another non-maximal symmetric equation.

Surfaces associated with first-order ODEs

A. J. Pan-Collantes[1], *J. A. Álvarez-García*[2]

[1] *Department of Mathematics, IES San Juan de Dios, Medina-Sidonia, Spain*; antonio.pan@uca.es

[2] *Department of Mathematics, IES Jorge Juan, San Fernando, Spain*; jose.alvg@gmail.com

A link between first-order ordinary differential equations (ODEs) and 2-dimensional Riemannian manifolds is explored. Let U be an open subset of $\mathbb{R}^2$ with coordinates (x, u), and consider a first-order ODE

$$\frac{du}{dx} = \phi(x, u), \tag{1}$$

where ϕ is a smooth function defined on U. We associate with equation (1) a Riemannian metric g defined on U, and explore some properties of the resulting surface (U, g). We have found a connection between Jacobi fields of this surface and Lie point symmetries of the ODE. Also, it is proven that if the corresponding surface is flat (zero Gaussian curvature), then the ODE can be integrated by quadratures.

Next, we investigate deformations of the surface (U, g). We consider Riemannian metrics $\{g_\epsilon\}$ defined on U, indexed by a smooth function $\epsilon \in \mathcal{C}^\infty(U)$, and such that $g_0 = g$. This deformation is defined to preserve specific features of g that are relevant to the ODE.

We establish a relationship between certain Jacobi type fields on the deformed surface and the integrability of the ODE, and we show that there is a class of vector fields, beyond Lie point symmetries, which are useful for solving first-order ODEs. As a result, it is concluded that the deformation into a constant curvature surface leads to the integrability by quadratures of the given ODE. In particular, the deformation into a zero curvature surface corresponds to the finding of an integrating factor.

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Construction of solutions of analogs of the Schrodinger time equations corresponding to the Hamiltonian system H^{3+1+1} of Kimura

V. A. Pavlenko

Institute of Mathematics with Computing Centre - Subdivision of the Ufa Federal Research Centre of Russian Academy of Science, Ufa, Russia; mail@pavlenko.school

A pair of joint solutions of analogs of time Schrodinger equations which defined by the Hamiltonians of $H^{3+1+1}_{s_k}(s_1, s_2, q_1, q_2, p_1, p_2)(k = 1, 2)$ of the Hamiltonian system H^{3+1+1}, which is written out in Kimura's article [1] are built. These analogues of the Schrodinger equations are linear evolutionary equations with times s_1 and s_2, each of which depends on two spatial variables.

2×2 matrix joint solutions of scalar linear evolutionary equations are constructed $\Psi'_{s_k} = H^{3+1+1}_{s_k}(s_1, s_2, x_1, x_2, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\Psi$ with times s_1 and s_2, which are analogs of the Schrodinger time equations. These equations correspond to the Hamiltonian system H^{3+1+1}, being a representative of the hierarchy of degenerations of the isomonodromic Garnier system described by H. Kimura (see [1]) in 1986. The constructed solutions are explicitly expressed in terms of joint solutions of matrix linear pairs of IDM from the article [2]. The condition for the compatibility of such pairs is just the Hamiltonian ODES corresponding to the Hamiltonians of the system H^{3+1+1}. A replacement linking matrix solutions of analogs of the Schrodinger time equations defined by two forms (rational and polynomial in coordinates) of the H^{3+1+1} system is given. This substitution is a quantum analogue of the well-known canonical transformation connecting the Hamiltonian equations of the system H^{3+1+1} in two given forms.

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Analytical Study of Nerve Axons

Z. Pinar Izgi

Department of Mathematics, Tekirdağ Namık Kemal University, Tekirdağ, Turkey;
zpinar@nku.edu.tr

Nerve axon is a biophysical process that is complex and is affected by physical, chemical, and thermal changes [1]. A multidiscipline study is needed to understand the phenomena. Historically, Hodgkin-Huxley model [2, 3], Fitz-Hugh-Nagumo model [4], etc. are the most known which are focused on the electrical part of the nerve pulse. However, the morphology of axons, whether they are myelinated or not, is important for pulse propagation. We will focus on the model that simulates the mechanical wave expressing the deformation in the axon wall, considering the axon morphology and electrical structure so it is assumed as biomembrane. HeimburgJackson [5, 6, 7, 8] proposed the model that is a Boussinesq-type equation. Considering the analytical methods, explicit solutions of the model are proposed and the changes in wave profile are seen clearly.

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Helical flows and their two-dimensional analogs

V. V. Pukhnachev[1,2]

[1]*Lavrentyev Institute of Hydrodynamics*; pukhnachev@gmail.com
[2]*Novosibirsk State University, Novosibirsk, Russia*;

Helical flows are characterized by the alignment of the velocity vector and its curl vector. This class of flows was initially discovered by I.S. Gromeka in 1981 and independently by E. Belrami in 1989. Gromeka and Beltrami described stationary helical flows in ideal incompressible fluids. In 1896, V.A. Steklov identified helical flows in viscous fluids, which are inherently non-stationary.

One remarkable property of helical flows is the presence of the Bernoulli integral, even though the fluid motion is non-potential. This presentation contains known examples of helical flows obtained by O.A. Bogoyabvlenskij, V.A. Galkin, G.B. Sizykh, and others. Additionally, new solutions describing plane and axi-symmetric analogs of helical flows will be presented. Lastly, solutions of this kind for second-order fluids will be discussed.

Some Consequences of the Connection Between Singularity Analysis and Symmetry Analysis

Asghar Qadir[1], *Adnan Aslam*[2], *Hina M. Dutt*[3]

[1]*Pakistan Academy of Sciences, 3 Constitution Avenue, Islamabad, Pakistan*; asgharqadir46@gmail.com

[2]*School of Electrical Engineering and Computer Science, National University of Sciences & Technology, H-12 Islamabad, Pakistan*; adnan.aslam@seecs.edu.pk

[3]*School of Electrical Engineering and Computer Science, National University of Sciences & Technology, H-12 Islamabad, Pakistan*; hina.dutt@seecs.edu.pk

A connection had been noted by one of us (AQ) between Singularity Analysis and Lie Symmetry Analysis, in that both provide [1] remarkable new results arising from singularities in the complex plane for each independent variable. For the first it was used directly. For the second it arose from the use of complex methods for Symmetry Analysis. That work had focused on the case of one independent variable. It was pointed out that for ordinary differential equations Painlevé analysis provides classes on the basis of the nature of the singularities, while Lie Symmetry Analysis provides classes on the basis of the infinitesimal symmetries of the differential equations. As such, it would be worthwhile to explore the symmetry classification of the Painlevé classes to obtain the solutions for the Painlevé classes directly. This is done here.

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Optimal system and classification of invariant solutions of a nonlinear class of wave equations and their conservation laws

Ali Raza

Centre For Mathematics and Statistical Sciences, Lahore School of Economics, Pakistan;
raza.ali@sms.edu.pk

We study the classication of invariant solutions of a class of nonlinear wave equations using Lie symmetry analysis and the underlying optimal systems of subalgebras. We propose a classication of Lie generators via optimal systems for four cases that arise therein. These optimal systems are presented in a convenient tree leaf diagram. Corresponding to each class, complete symmetry reductions and the invariant solutions are presented. To the best of our knowledge, this

classication of optimal systems is new and do not appear in the literature. Our results also lead to the establishment of the local conservation laws corresponding to each conserved vector via the multiplier approach.

Integration by quadratures of Lie–Hamilton systems

R. Campoamor-Stursberg

Instituto de Matemática Interdisciplinar and Dpto. Geometría y Topología, UCM, Madrid, Spain; rutwig@ucm.es

Lie–Hamilton systems are characterized by the property of admitting a nonlinear superposition principle, that enables to describe the general solution in terms of a certain number of particular solutions and significant constants. The additional geometrical compatibility structure allows a systematic construction of the constants of the motion. However, depending on the corresponding Vessiot–Guldberg algebra, as well as a suitable realization in terms of vector fields, LH systems can also be analyzed for their integrability (by quadratures), using the Lie symmetry method and local diffeomorphisms. The classification of LH systems on the real plane is revisited from this point of view, determining conditions that enable us to find the general solution without need of superposition rules. By means of examples, the problem of the integrability of their quantum deformations, that do no more formally correspond to LH systems, is also explored.

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Solutions to the Wave Equation With Non-constant Speed Through the Method of Differential Constraints

A. Rizzo

MIFT, Università degli Studi di Messina, Italy; alerizzo@unime.it.

The idea of commuting flows plays a key role in the theory of Hamiltonian Systems. In fact, if a Hamiltonian System admits infinitely many commuting flows then it has the integrability property. It is known that 2 components Hamiltonian quasilinear systems , i.e. hydrodynamic type systems of the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} h_u \\ h_v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} f_u \\ f_v \end{pmatrix} \tag{1}$$

commute if and only if

$$h_{uu} f_{vv} - h_{vv} f_{uu} = 0. \tag{2}$$

Avoiding trivial cases, the previous equation can be rewritten under the form of the wave equation with non constant speed

$$f_{vv} - a^2(u, v) f_{uu} = 0, \tag{3}$$

where we set $a^2(u, v) = \dfrac{h_{vv}}{h_{uu}}$.

Motivated by this viewpoint, we apply the reduction procedure of differential constraints to obtain a complete set of solutions of such an equation for some fixed velocities $a^2(u, v)$. As a result, we present some examples of Hamiltonian integrable systems (as the shallow water equations) with relative symmetries, conserved quantities and solutions.

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Stability of two-sided estimates of differential equations sets solutions with disturbances

A. N. Rogalev

Institute of Computational Mathematics, SBRAS, Krasnoyarsk, Russia;
rogalyov@icm.krasn.ru

The paper studies algorithms for solving applied problems of estimating the domains of solutions of differential equations. Such applied problems arise when estimating the practical stability of movements over a finite time interval, when determining the reachability areas of controlled systems, when estimating the survivability areas of controlled systems, when calculating guaranteed (non-probabilistic) boundaries of zones of dangerous states of technical systems, when calculating threshold values of parameters of technical systems, which correspond to the boundaries of dangerous zones, when computing the maximum deviations of movements to control the entry of the system trajectory into the dangerous zone. In the listed problems, only estimates of their ranges of values are known for some of the parameters. Then the sets of solutions to the ODE system with perturbing (controlling) actions are determined

$$\frac{dy}{dt} = f(t, y, u), u \in U, y_0 \in Y_0, \tag{1}$$

where $y-$ is the $n-$ dimensional phase vector of the system in the Euclidean space R^n, $u-$ the vector of perturbing (controlling) actions, $U-$ the compact in the Euclidean space R^p , and Y_0 the initial data areas. This leads to the emergence of sets of solutions. The set of solutions to problem (1) depending on the perturbing parameter

$$Y(t) = Y(t, Y_0, U) = \bigcup_{y_0 \in Y_0} y(t, y_0, U) = \{y : y(t_0) \in Y_0, \forall t \geq 0, \frac{dy}{dt} = f(t, y, u)\} \tag{2}$$

is called a tube of trajectories and, in the general case, forms a rather complicated structure.

The study of the set of solutions (2) is complicated by the fact that, in addition to stable solutions of system (2) , the set of trajectories may contain singular points, separatrices, and limit cycles. All this is the reason for the difficult to predict behavior of the trajectories of the solution sets. For this and other reasons, this set, in most cases, cannot be calculated. Instead, we compute approximations to this set that include a set of solutions. The complexity of solving these problems lies in the fact that most methods for estimating the sets of solutions of ODE systems (or computing the upper and lower bounds of the solutions) lead to a strong growth of the boundaries of these sets of solutions.

To eliminate most of the difficulties in implementing methods for estimating sets of solutions, it is proposed to use symbolic formulas - these are either combinations of signs of operations, constants and variables that have an independent meaning . These formulas indicate an algorithm for computing the value of an expression, or a record of an expression to determine the range of values of a value through its parameters. It is useful to regularize the estimates of the boundaries of the solution sets by passing to a linear approximation of the original system. This helps to overcome in some cases the growth of solution set boundaries [1], [2]. Nontheless, when constructing two-sided estimates (inclusions) of sets of solutions to ODE systems, changes appear in the coefficients of the ODE system, changes in the dimension of the problem, and some other changes. The purpose of the article is to establish which of the reasons or combinations of reasons are the basis for the increase in the boundaries of estimates.

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Regularizing factors for the Euler-Poisson equations

O. S. Rozanova

Lomonosov Moscow State University, Russia; rozanova@mech.math.msu.su

We consider the Euler-Poisson equations

$$\begin{aligned} \frac{\partial n}{\partial t} &+ \operatorname{div}(n\mathbf{V}) = 0, \\ \frac{\partial \mathbf{V}}{\partial t} &+ (\mathbf{V}\cdot\nabla)\,\mathbf{V} = k\,\nabla\Phi + F_p + F_f + F_v, \\ \Delta\Phi &= n - n_0, \end{aligned}$$

where n is the density, $n_0 = \text{const} \geq 0$ is the density background, $\mathbf{V}$ is the velocity of particles, Φ is a force potential, $k = \text{const}$, $k > 0$ corresponds to the repulsive force (the models of plasma and semi-conductors), $k < 0$ corresponds to the attractive force (the models of astrophysics). Here F_p, F_f, F_v are pressure, friction, and viscosity, respectively, which, depending on the model, can be taken into account or neglected.

It is well known that the solution to the Cauchy problem for the Euler-Poisson system without any additional factors, as a rule, loses smoothness in a finite time; moreover, a strong delta singularity is formed in the density component.

Our main question is: can we delay, remove or weaken the singularity using pressure, friction or viscosity?

We consider the pressure term as $F_p = -\alpha\dfrac{\nabla p(n)}{n}$, $p(n) = \dfrac{1}{\gamma}n^\gamma$, $\gamma > 1$, $\alpha = \text{const} \geq 0$ friction term as $F_f = -\nu(n)\mathbf{V}$, $\nu(n) \geq 0$, and viscosity term as $F_v = \mu\Delta\mathbf{V}$, $\mu = \text{const} \geq 0$ and restrict ourselves to the case of a one-dimensional model in space.

We show that

- the pressure generally does not eliminate or delay a blowup, however changes the type of singularity to a weaker one;
- Constant friction delays the formation of a singularity at fixed initial data; for fixed initial data, one can choose a friction coefficient large enough to guarantee global smoothness of the solution, but for arbitrarily strong constant friction, one can find initial data that generates a finite-time singularity [1];
- one can find $\nu(n)$ such that the solution remains smooth for all $t > 0$ [2];
- an arbitrary small constant viscosity guarantees global smoothness of the solution;
- There are examples of viscosity that depends on the solution and does not prevent the formation of a singularity.

We discuss also the influence of friction in the multidimensional case [3].

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λ-symmetries for the Levinson-Smith equation

A. Ruiz[1], *C. Muriel*[2]

Departamento de Matemáticas, Universidad de Cádiz, Puerto Real, Spain;
[1]adrian.ruiz@uca.es, [2]concepcion.muriel@uca.es

The Levinson-Smith equation $\ddot{x}+f(x,\dot{x})\dot{x}+g(x)=0$ was introduced in [1] as a generalization of the classical Liénard equation $\ddot{x}+f(x)\dot{x}+g(x)=0$. In the Levinson-Smith equation, the function $f(x,\dot{x})$ represents the damping term which, in contrast to the Liénard equation, can depend on the derivative $\dot{x}$. The qualitative behavior of solutions to the Levinson-Smith equation has been widely studied in the recent literature. For instance, some conditions under which the equation admits periodic solutions were established in [1, 2].

In this talk the theory of λ-symmetries [3] is applied to the Levinson-Smith equation with the goal of determining exact general solutions for some families of equations. As a result of the study performed, a family of equations that admits a λ-symmetry of a specific form is deduced. Moreover, the λ-symmetry-based integration method leads to the general exact solution of the family of equations in terms of the solution to a separable first-order equation. Remarkably, the obtained family includes some cases of the general Liénard-type equation, that appears often in Physics and Biology [4], and contains equations that only admit ∂_t as Lie point symmetry [5, 6] for which the classical Lie reduction method fails in providing the exact general solution.

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Homogenization of diffusion processes with singular drifts and potentials via unfolding method

Adisak Seesanea

School of Integrated Science and Innovation, Sirindhorn International Institute of Technology, Thammasat University, Pathum Thani, Thailand; adisak.see@siit.tu.ac.th

Abstract

We shall discuss a study of homogenization problems for elliptic equations of the form

$$\begin{cases} \mathfrak{L}_\delta u_\delta + \lambda u_\delta = f_\delta & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

where $\delta > 0$, $\lambda \in \mathbb{R}$, D is a bounded open set in $\mathbb{R}^d$, and $f_\delta \in H^{-1}(D)$. The operator $\mathfrak{L}_\delta u = -\text{div}\left(A^\delta \nabla u + C^\delta u\right) + B^\delta \nabla u + k^\delta u$ driven by uniformly bounded diffusion coefficients A^δ, where drifts B^δ, C^δ and potential k^δ are possibly unbounded. An application to homogenization of the corresponding diffusion processes will be demonstrated. This talk is based on joint work with Toshihiro Uemura (Kansai University).

KEYWORDS: Homogenization, diffusion processes, singular drifts, unfolding method

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Using Conservation Laws to Solving the Boundary Value Problems of Deformable Solid Mechanics

S. I. Senashov[1], *O. V. Gomonova*[2], *I. L. Savostyanova*[3]

[1]*Department of Information and Economic Systems, Reshetnev Siberian State University of Science and Technology, Krasnoyarsk, Russia*; sen@sibsau.ru

[2]*Department of Higher Mathematics, Reshetnev Siberian State University of Science and Technology, Krasnoyarsk, Russia*; gomonova@sibsau.ru

[3]*Department of Information and Economic Systems, Reshetnev Siberian State University of Science and Technology, Krasnoyarsk, Russia*; ruppa@inbox.ru

Equations of solid mechanics have been studied by the methods of group analysis for more than 50 years [1].

In this period, new classes of exact solutions were obtained and the structure of the studying equations was investigated. The exact solutions give possibility to solve the boundary value problems by semi-inverse methods.

Solving of boundary value problems of the deformable solid mechanics equations was started in the works [2, 3]. Investigations in this area revealed that the boundary value problems of systems of elliptic and hyperbolic equations generally can be solved effectively with the use of conservation laws (see [4, 5] and the references).

The present work is devoted to a review of the methods of solving of the boundary value problems for hyperbolic and elliptic systems of the deformable solid mechanics equations.

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Lie Group Analysis, Optimal System And Invariant Solutions Of Jeffery-Hamel Flow Equation

Rajeswari Seshadri

Department of Mathematics, Pondicherry University, Kalapet 605014, India;
seshadrirajeswari@pondiuni.ac.in

In fluid dynamics, a flow created by a converging or diverging channel with a source or sink at the point of intersection of the two plane walls is represented by the Jeffery-Hamel (J-H) flow equation. The objective of this analysis is to obtain possible analytical solutions of the governing equation. The point symmetries, the optimal system of one-dimensional subalgebra, and the associated similarity-reductions are derived using the Lie group analysis. Painlevé analysis is also carried out to study the singularity structure of its solution. A truncated Painlevé series solution has been derived as the general solution of the JH equatioon. Though several aspects of J-H flow and its numerical solutions are studied, the Lie symmetry and the singularity analysis is done for the first time.

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Solution of a boundary value problem to ordinary differential equations by the least squares collocation method and multipoint Padé approximation

V. P. Shapeev

Khristianovich Institute of Theoretical and Applied Mechanics SB RAS, Novosibirsk, Russia; shapeev.vasily@mail.ru

It is known that the Padé approximation (PA) has increased accuracy compared to the polynomial approximation. Since the discovery of this algorithm by Henri Padé, the problems of using PA for the approximate representation of one-dimensional functions have attracted the attention of many mathematicians. Applications of PA to solving integral and differential equations are still much smaller in number compared to cases where various polynomials are used for this purpose. At the moment, for objective reasons, there are practically no publications on the application of AP to solving the boundary value problem for ODE. In this paper, a new numerical algorithm for solving this problem using PA in the form of [L/M] with undetermined coefficients is proposed and implemented. As a result of multipoint approximation of a given problem, as in the case of solving the Cauchy problem for ODE, a system of nonlinear equations with respect to its coefficients is obtained. After preliminary partial linearization, an iterative process for solving the resulting equations is proposed, in which at each next iteration the values of part of the expressions in the equations are taken from the previous iteration so that at the next iteration the system of linear algebraic equations (SLAE) is solved. To limit the number of iterations, in particular, it is necessary to satisfy the condition that the residual functional of all equations after substituting into them the approximate solution obtained at the next iteration will not become less than a given small value.

To implement the algorithm, a program was written in the language of the Wolfram Mathematica system. The algorithm and program allow the setting of two conditions at any point in the solution area, including the simultaneous setting of the value of the solution and its derivative at one point if a solution to such a problem exists. The individual properties of the new algorithm and the quantitative characteristics of the solutions to the problem obtained using it are studied using the example of specific equations. In the results obtained for solving test examples and examples taken from the literature, attention was paid to the achievable accuracy of solutions, to the dependence of the number of iterations on the conditionality of the resulting SLAEs and their dependence on the complexity of the behavior in the area of solving the problem of functions that determine the type of equation and, accordingly, the solution itself. In cases of relatively simple equations when using arithmetic on a computer with numbers in double format, the error in solutions was close to the error in rounding numbers. In this case, the iterations converged from the initial approximation in the form of a constant specified in the entire domain of solution of the problem. In problems with nonlinear equations, preliminary linearization of the equation was used.

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Existence and regularity for a class of double phase parabolic problems

R. Arora[1], *S. Shmarev*[2]

[1]*Department of Mathematical Sciences, Indian Institute of Technology (IIT-BHU), Varanasi, 221005, Uttar 665 Pradesh, India*; rakesh.mat@iitbhu.ac.in

[2]*Department of Mathematics, University of Oviedo, c/Federico García Lorca, Oviedo, 33007, Asturias, Spain*; shmarev@uniovi.es

We study the homogeneous Dirichlet problem for the equation

$$u_t - \operatorname{div}\left(\mathcal{F}(z, \nabla u)\nabla u\right) = f, \quad z = (x, t) \in Q_T = \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with the C^2 boundary, and the flux function has the form

$$\mathcal{F}(z, \xi) = a(z)|\xi|^{p(z)-2} + b(z)|\xi|^{q(z)-2}.$$

The variable exponents p, q and the nonnegative modulating coefficients a, b are given Lipschitz-continuous functions. It is assumed that the exponents satisfy the inequality $\frac{2N}{N+2} < p(z), q(z)$ and the balance condition

$$|p(z) - q(z)| < \frac{2}{N+2} \text{ in } \overline{Q}_T.$$

The coefficients a, b need not be strictly positive and may vanish on parts of the problem domain, but it is assumed that

$$a(z) + b(z) \geq \alpha \quad \text{in } \overline{Q}_T$$

with some constant $\alpha > 0$. The term “double phase” reflects the fact that the properties of the flux $\mathcal{F}$ change according to behaviour of the modulating coefficients $a(z)$ and $b(z)$. Nonlinear operators of this type were introduced in 70-80th by J.Ball and V.V.Zhikov in the context of the nonlinear elasticity theory.

We find conditions on the source f and the initial data $u(\cdot, 0)$ that guarantee the existence of a unique strong solution u with $u_t \in L^2(Q_T)$ and $a|\nabla u|^p + b|\nabla u|^q \in L^\infty(0, T; L^1(\Omega))$. The solution possesses the property of global higher integrability of the gradient,

$$|\nabla u|^{\min\{p(z),q(z)\}+r} \in L^1(Q_T) \quad \text{with any } r \in \left(0, \frac{4}{N+2}\right),$$

which is derived with the help of new interpolation inequalities in the variable Sobolev spaces. The global second-order differentiability of the strong solution is proven:

$$D_i\left(\sqrt{\mathcal{F}(z, \nabla u)}D_j u\right) \in L^2(Q_T), \quad i, j = 1, 2, \ldots, N.$$

The analytical framework for the proof of existence is furnished by the theory of the Musielak-Orlicz-Sobolev spaces. The detailed proofs were published in [1, 2].

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Dirichlet problems involving measures and several sublinear terms: fixed point theory approach

Toe Toe Shwe[1] *and Adisak Seesanea*[2]

School of Integrated Science and Innovation, Sirindhorn International Institute of Technology, Thammasat University, Pathum Thani, Thailand;
[1]m6522040820@g.siit.tu.ac.th, [2]adisak.see@siit.tu.ac.th

This work illustrates a method in fixed point theory for the existence of continuous solutions to Dirichlet problems of the type

$$\begin{cases} -\Delta u &= \sum\limits_{i=1}^{m} \sigma_i u^{q_i} + \sigma_0, \quad u > 0 \qquad \text{in} \quad \Omega, \\ \lim\limits_{x \to y} u(x) &= f(y) \qquad\qquad\qquad\qquad y \in \partial^\infty \Omega, \end{cases}$$

in the sublinear case $0 < q_i < 1$, where each coefficient σ_i are nonnegative Radon measures in a regular domain $\Omega \subset \mathbb{R}^n$ which possesses the positive Green function, and f is a nonnegative continuous function on $\partial^\infty \Omega$. Uniqueness and pointwise estimates of such solutions are also discussed.

KEYWORDS: Continuous solution, Schauder fixed point theorem, two-sided pointwise estimates, measure data

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Exact Solutions of Non-Linear Partial Differential Equations Using Interesting Analytical Methods

Sekson Sirisubtawee

Department of Mathematics, Faculty of Applied Science
King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand;
sekson.s@sci.kmutnb.ac.th

There are now many analytical methods that have been used to construct exact solutions of nonlinear partial differential equations (PDEs) or systems of nonlinear PDEs. These methods are mainly based on first converting the original PDE or system of PDEs into a nonlinear ordinary differential equation (ODE) or system of nonlinear ODEs via a complex traveling wave transformation and the chain rule. Next, some special techniques are applied to the ODEs in order to derive their exact solutions and then these exact ODE solutions are transformed back into the exact solutions of the PDEs. Some examples of the useful analytical methods that have been employed for deriving exact solutions of the PDEs are the $\left(G'/G, 1/G\right)$-expansion method, the modified Kudryashov method, the Exp-function method, the auxiliary equation method and the Sardar sub-equation method. With these analytical methods there is no further demand for the normalization or discretization in the calculation process that are often required by numerical schemes. The aim of this talk is to introduce the preliminary concepts of some of these analytical methods that have been used to obtain exact solutions for some PDEs arising in important physics and engineering problems. In addition, some advantages of the described methods will be clarified.

Group Classification of Heat and Mass Transfer Equations

S. V. Meleshko[1], *I. V. Stepanova*[2]

[1]*School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima, Thailand*; sergey@math.sut.ac.th

[2]*Department of Differential Equations in Mechanics, Institute of Computational Modelling of Siberan Branch of Russian Academy of Sciences, Krasnoyarsk, Russia*; stepiv@icm.krasn.ru

We consider nonlinear heat and mass transfer equations taking into account that coefficients of thermal conductivity κ, diffusion D and parameters concerning to Soret and Dufour effects (D^F and D^θ) depend on the desired functions of temperature T and concentration of the light component in the liquid solution C. The described system has the form

$$\begin{aligned} \frac{\partial T}{\partial t} &= \operatorname{div}\Big(\kappa(T,C)\nabla T + D^F(T,C)\nabla C\Big),\\ \frac{\partial C}{\partial t} &= \operatorname{div}\Big(D(T,C)\nabla C + D^\theta(T,C)\nabla T\Big). \end{aligned} \tag{1}$$

Here t means the time, the generator ∇ is calculated with respect to three space coordinates x^i, $i = 1,2,3$. It is necessary to note that class of equations (1) contain some simple equations, the group properties of which were treated earlier. For example, the group classification of nonlinear heat transfer equation with respect to conductivity coefficient was carried out by L. V. Ovsyannikov [1].

The structure of the basic Lie algebra L^0 of class (1) is quite simple

$$L^0 = \Big\langle \partial_t,\ \partial_{x^i},\ 2t\partial_t + \sum_{i=1}^{3} x^i\partial_{x^i},\ x^j\partial_{x^i} - x^i\partial_{x^j} \Big\rangle,\ i,j = 1,2,3,\ i \neq j.$$

It contains four translation operators, one dilation operator which is not associated with the unknown functions T and C, and three rotation operators. All the obtained operators are natural and are admitted by many mathematical models of continuum mechanics.

The problem of group classification of system (1) was solved with respect to variable transport coefficients in the cases when $D^\theta \equiv 0$ (see [2]) or $D^F \equiv 0$ (see [3]). But the group properties of system (1) where the both coefficients do not vanish are interesting not only as extracting information on quality properties of the system but also as check of the Onsager's reciprocity relation [4]. The latter predicates the coefficients D^F and D^θ should be closely related.

The present work is devoted to group classification of full class (1) with respect to four transport coefficients. Some conclusion concerning to group properties of the equations are derived depending on the obtained classifying equations and forms of the classified functions.

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Interactions of Solitons and Lumps in the Cylindrical Kadomtsev–Petviashvili Equation

Zh. Zhang[1], *W. Hu*[2], *Q. Guo*[1], *Y. Stepanyants*[3]

[1]*Guangdong Provincial Key Laboratory of Nanophotonic Functional Materials and Devices, South China Normal University, Guangzhou 510631, P. R. China*; 1820210058@qq.com; guoq@scnu.edu.cn

[2]*College of Physics and Optoelectronic Engineering, Zhongyuan University of Technology, Zhengzhou 450007, China*; phyhwc@163.com

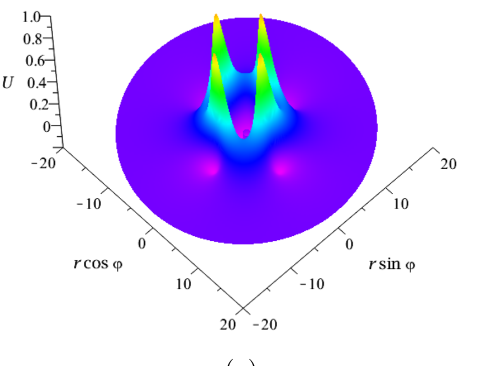
[3]*School of Mathematics, Physics and Computer Sciences, University of Southern Queensland, Toowoomba, QLD, 4350, Australia*; Yury.Stepanyants@unisq.edu.au

We study solitary waves in the cylindrical Kadomtsev–Petviashvili equation designated to media with positive dispersion (the cKP1 equation):

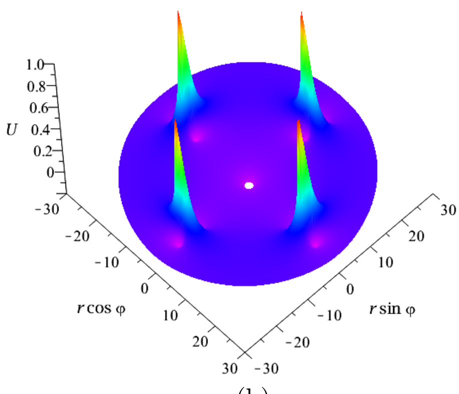
$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}+\frac{1}{c}\frac{\partial u}{\partial t}-\frac{\alpha}{c}u\frac{\partial u}{\partial t}-\frac{\beta}{2c^5}\frac{\partial^3 u}{\partial t^3}+\frac{u}{2r}\right)=\frac{c}{2r^2}\frac{\partial^2 u}{\partial \varphi^2}, \tag{1}$$

where $u(t,r,\varphi)$ is a wave perturbation that depends on time t and two spatial coordinates in the cylindrical coordinate frame (r,φ), c is the speed of long linear waves, α and β are the coefficient of nonlinearity and dispersion, respectively, which depend on parameters of a particular physical problem. Equation similar to that was derived for the first time for surface water waves in a shallow basin, and then, for internal waves, and for plasma waves.

By means of the Darboux–Matveev transform, we derive exact solutions that describe two-dimensional solitary waves (lumps), lump chains, and their interactions. One of the obtained solutions describes the modulation instability of outgoing ring solitons and their disintegration onto a number of lumps as shown in the figure. The figure shows the expansion of four lumps at $t = 6$ (a) and at $t = 30$ (b) that emerge in the result of modulation instability.



(a)



(b)

We also present solutions describing decaying lumps and lump chains of a complex spatial structure – ripplons. Then, we study normal and anomalous (resonant) interactions of lump chains with each other and with ring solitons. Results obtained agree with the data of numerical modelling.

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On the Generalized Korteweg-de Vries equations with time-dependent coefficients

D. A. Suchkova[1], *F. S. Nasyrov*[2]

Department of Mathematics, Ufa University of Science and Technology
Ufa, Russian Federation;
[1]dil16ara@yandex.ru, [2]farsagit@yandex.ru

The Generalized Korteweg – de Vries equation (GKdV) with noise $V(t)$ can be formally written as:

$$u_t + (f(u))_x V'(t) + u_{xxx} = 0. \tag{1}$$

We will be interested in the Cauchy problem for GKdV with time-dependent coefficients, which can be conveniently interpreted as noise. By *noise* we mean an arbitrary continuous, possibly non-differentiable function $V(t)$, $V(0) = 0$, or a random process $V(t)$, $V(0) = 0$, with continuous paths. In particular, these can be a Wiener process or a fractal Brownian motion.

Let us formulate the problem. Since the solution $u(x,t)$ depends on noise, it is necessary to look for a solution of (1) as a function $u(x,t) \equiv u(x,t,V(t))$. Using the technique of symmetric integrals [1], equation (1) will be written in integral form:

$$\int_0^t [u_s(x,s,V(s)) + u_{xxx}(x,s,V(s))]ds + \int_0^t [u_v(x,s,V(s)) + (f(u(x,s,V(s))))_x] * dV(s) = 0. \tag{2}$$

It is shown that solving equation (2) reduces to solving the system:

$$\begin{cases} u_t(x,t,v)|_{v=V(t)} + u_{xxx}(x,t,V(t)) = 0, \\ u_v(x,t,v)|_{v=V(t)} + (f(u(x,t,V(t))))_x = 0. \end{cases}$$

Theorem. *Let $V(t)$, $t \in [0,T]$, $V(0) = 0$, be a continuous function. Then the function $u(x,t,V(t)) = (-3t)^{-\frac{1}{3}} \int\limits_{-\infty}^{+\infty} Ai\left(\frac{x-y}{(-3t)^{\frac{1}{3}}}\right) \varphi(y,V(t))\, dy$ is the solution to the equation GKdV (2) with initial condition $\varphi(x,v) = u(x,0,v)$, where $Ai(z) = \frac{1}{\pi}\int\limits_0^{\infty} \cos\left(\frac{y^3}{3} + yz\right) dy$ is the function of Airy [3].*

Remark. The talk will also present KdV solutions with noise affecting the dispersion term, the dispersion and nonlinear terms [6].

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On the Hierarchy of Differential-Invariant Solutions

A. A. Talyshev

Department of Mechanics and Mathematics, Novosibirsk State University, Novosibirsk, Russia; tal@academ.org

Differential-invariant solutions are a generalization of invariant and partially invariant solutions [1]. Each differentially-invariant solution is characterized by the sequence dimensions of orbits $d_0, d_1, \ldots$ of this solution in extended spaces [2], [3], [4], where d_j is the dimension of the orbit of the solution in the j-extended space.

In the works [1] and [5] described the construction algorithm optimal system of subalgebras. Then this optimal system of subalgebras used to construct invariant and partially invariant solutions.

Here we consider a similar question for differential-invariant solutions.

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Exact Solutions of Nonlinear Partial Differential Equations. Traveling Wave Propagating in Murnaghan's Model for an Isotropic Elastic Medium and in Rigid Thermal Conductors

M. Tantawy

October 6 University, Egypt; mtantawymath@gmail.com

We present a summary of the unified method UM for finding exact solutions to Partial Differential Equations PDEs [1, 2]. The solutions are classified as polynomial and rational function solutions with auxiliary functions that satisfy appropriate ordinary differential equation ODEs. Here we applied UM to find family of traveling wave solutions TWS for Murnaghan's model for an isotropic elastic medium [3]. Here, attention is focused on the case when the speed of the moving frame is equal to the wave speed. Computational are derived to show certain critical values of the nonlinearity and double-double dispersion of the model. Also, we applied for getting TWS of one-dimensional of nonlinear equations of extended thermodynamics for temperature and heat flux in an infinite rigid thermal conductor [4]. The behavior of these solutions is investigated for the temperature and the heat flux with the material properties.

Keywords: Unified Method; Traveling wave solutions; Murnaghan's model; temperature; Heat flux

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Finite-dimensional dynamics of evolutionary systems with several spatial variables

Tao S.

Lomonosov Moscow State University, Moscow, Russia; 86taosinian@gmail.com

We have evolutionary equation systems such as following:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \ldots, \frac{\partial^k \mathbf{u}}{\partial \mathbf{x}^k}\right).$$

The above system generates a flow on the maximal integral manifolds associated with certain completely integrable distributions P (see[1,2]), i.e.

Now assume we have an overdetermined system of partial differential equations as following, which can generate the distribution:

$$\frac{\partial^{q+1}\mathbf{v}}{\partial \mathbf{x}^{\sigma+1_i}} = \mathbf{V}_{\sigma+1_i}\left(\mathbf{x}, \mathbf{v}, \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \ldots, \frac{\partial^q \mathbf{v}}{\partial \mathbf{x}^q}\right), \quad |\sigma| = \sigma_1 + \cdots + \sigma_n = q; i = 1, \ldots n.$$

Let S be a shuffling symmetry of the distribution P [3]. There are a unique set of functions $\varphi^1, \ldots, \varphi^m$ on J^q such that

$$S = \sum_{j=1}^{m} \varphi^j \frac{\partial}{\partial v_o^j} + \sum_{\substack{|\sigma|=1 \\ j=1,\ldots,m}} \mathcal{D}^\sigma\left(\varphi^j\right) \frac{\partial}{\partial v_\sigma^j} + \cdots + \sum_{\substack{|\sigma|=q \\ j=1,\ldots,m}} \mathcal{D}^\sigma\left(\varphi^j\right) \frac{\partial}{\partial v_\sigma^j}.$$

Here $o = (0, \ldots, 0)$ is zero multi-index, $\mathcal{D}^\sigma = \mathcal{D}_1^{\sigma_1} \circ \cdots \circ \mathcal{D}_n^{\sigma_n}$, and $\mathcal{D}_i^s$ is the s-th degree of the operator

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \sum_{\substack{0 \le |\sigma| \le q \\ j=1,\ldots,m}} v_{\sigma+1_i}^j \frac{\partial}{\partial v_\sigma^j} + \sum_{\substack{0 \le |\sigma| = q \\ j=1,\ldots,m}} V_{\sigma+1_i}^j\left(\mathbf{x}, \mathbf{v}_\sigma\right) \frac{\partial}{\partial v_\sigma^j} \quad (i = 1, \ldots n)$$

Note that the distribution P is generated by the vector fields $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

To obtain a solution to the evolutionary system, we perform shifts along the vector field S. This vector field is determined by solving the system, and it allows us to find solutions to the overdetermined system.

The examples of the Boussinesq equation will be used to illustrate this method

$$\begin{cases} u_t = u_{xx} + 2v_x, \\ v_t = -v_{xx} + 2uu_x - 2u_y. \end{cases}$$

By aboving methods a family of exact solutions of the Boussinesq equation, dependent on six arbitrary parameters $\delta, \eta, C_1, \ldots, C_4$ and one arbitrary function $g(y)$ was constructed.

Note that the method of finite-dimensional dynamics for scalar evolution equations with one spatial variable was proposed in [4].

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Spectrum of a problem about the flow of a polymeric viscoelastic fluid in a cylindrical channel (Vinogradov-Pokrovski model)

D. L. Tkachev[1], *E. A. Biberdorf*[2]

[1]*Sobolev Institute of Mathematics, Novosibirsk, Russia*;
[1]tkachev@math.nsc.ru, [2]ermolova@math.nsc.ru

We study the linear stability of a resting state for flows of incompressible viscoelastic polymeric fluid in an infinite cylindrical channel in axisymetric perturbation class. We use structurally-phenomenological Vinogradov-Pokrovski model as our mathematical model [1, 2].

We formulate two equations that define the spectrum of the problem. Our numerical experiments show that with the growth of perturbations frequency along the channel axis there appear eigenvalues with positive real part for the radial velocity component of the first spectral equation. That guarantees linear Lyapunov instability of the resting state [3].

The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. FWNF-2022-0008).

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On the Global Solvability of One-Dimensional Boundary Value Problems for the Equations of Fluid Filtration in a Poroelastic Medium

M. A. Tokareva[1], *A. A. Papin*[2]

Institute of Mathematics and Information Technologies, Altai State Universiay, Barnaul, Russia; [1]tma25@mail.ru, [2]papin@math.asu.ru

This quasilinear combined type system describes the spatial unsteady nonisothermal motion of a compressible fluid in a viscoelastic medium. The model equations in the absence of phase transitions and taking into account the dependence of the medium parameters on porosity and temperature have the form [1] – [3]:

$$\tfrac{\partial(1-\phi)\rho_s}{\partial t} + div((1-\phi)\rho_s\vec{v}_s) = 0, \quad \tfrac{\partial(\rho_f\phi)}{\partial t} + div(\rho_f\phi\vec{v}_f) = 0, \tag{1}$$

$$\phi(\vec{v}_f - \vec{v}_s) = -\frac{K(\phi)}{\mu(\theta)}(\nabla p_f + \rho_f\vec{g}), \tag{2}$$

$$\nabla\cdot\vec{v}_s = -a_1(\phi)\xi_1(\theta)p_e - a_2(\phi)\xi_2(\theta)(\frac{\partial p_e}{\partial t} + \vec{v}_s\cdot\nabla p_e), \tag{3}$$

$$\nabla\cdot\sigma + \rho_{tot}\vec{g} = 0, \quad \rho_{tot} = \phi\rho_f + (1-\phi)\rho_s, \tag{4}$$

$$(\rho_f c_f\phi + \rho_s c_s(1-\phi))\frac{\partial\theta}{\partial t} + (\rho_f c_f\phi\vec{v}_f + \rho_s c_s(1-\phi)\vec{v}_s)\nabla\theta = div(\lambda\nabla\theta), \tag{5}$$

$$p_{tot} = \phi p_f + (1-\phi)p_s, p_e = (1-\phi)(p_s - p_f), \tag{6}$$

where ϕ is porosity, $\vec{v}_f, \vec{v}_s$ are velocities of fluid and porous skeleton respectively, ρ_f, ρ_s are densities of fluid and solid phase respectively, p_s, p_f are pressure of solid and fluid phase respectively, p_e is the effective dynamic pressure, p_{tot} is the total pressure, ρ_{tot} is density of the two-phase medium, σ is the total stress tensor, $K(\phi), \mu(\theta)$ are the permeability and the fluid dynamic viscosity, θ is the temperature of the medium (the same for each phases), c_s and c_f are heat capacities for at constant volume of phases, $a_1(\phi), a_2(\phi)$, $\xi_1(\theta), \xi_2(\theta)$ are parameters of poroelastic medium, (x_1, x_2, x_3, t) are Eulerian coordinates. The thermal conductivity coefficient of the medium $\lambda(\phi)$ is taken in the form $\lambda(\phi) = \lambda_f\phi + \lambda_s(1-\phi)$, where λ_f, λ_s are the thermal conductivity of liquid and solid phase (averaged thermal conductivity).

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Three Dimensional Magnetostatic Atmospheres

G. M. Webb[1], *S. Nayak*[2], *Q. Hu*[3], *G. Li*[4]

[1]*CSPAR, The University of Alabama in Huntsville, Huntsville AL USA*; gmw0002@uah.edu
[2]*CSPAR, The University of Alabama in Huntsville, Huntsville AL, USA*; sn0073@uah.edu
[3]*Department of Space Sciences and CSPAR, The University of Alabama in Huntsville, Huntsville AL USA*; qiang.hu@uah.edu
[4]*Department of Space Sciences and CSPAR, The University of Alabama in Huntsville, Huntsville AL USA*; gang.li@uah.edu

We describe the work of [1], [2] and [3] on three dimensional structures of solar magnetostatic atmospheres, in which the $\mathbf{J}\times\mathbf{B}$ force on the plasma is balanced by the pressure gradient force and by the gravitational force. We study the particular model which assumes planar Cartesian geometry, in which the gravitational force is along the z-axis. The vertical current along the z-axis: $J_z = (\nabla\times\mathbf{B})_z/\mu_0$ is set equal to zero. From the conditions that the total pressure $p+B^2/(2\mu_0)$ should have continuous second order spatial partial derivatives one obtains integrals of the equations describing the equilibrium. The system of force equations is supplemented by Gauss's law $\nabla\cdot\mathbf{B}=0$. Gauss's law reduces to a second order, elliptic partial differential equation involving two scalar potentials $\phi(x,y,z)$ and $\psi(z,\partial\phi/\partial z)$. We provide a Lie symmetry group analysis (e.g. [5],[6]) of the elliptic partial differential equation for ϕ and ψ for the case of isothermal atmospheric structures that are periodic in x and y. We also discuss analogous structures for solutions involving spherical polar coordinates (r,θ,ϕ) in which the gravitational force is radially downward. More general magnetostatic atmospheres can be obtained by using numerical methods and including centrifugal forces for rotating atmospheres (e.g. [4]). We restrict our attention to analytical solutions.

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Enhancing Traffic Dynamics: Pioneering Applications of Fractional Differential Equations in Complex Systems

Benchawan Wiwatanapataphee

School of Electrical Engineering, Computing and Mathematical Science
Curtin University, Perth 6102, Australia; b.wiwatanapataphee@curtin.edu.au

This research explores the advancement and deeper understanding of fractional differential equations, which are key in modelling and interpreting complex physical phenomena, particularly in the realm of traffic flow, beyond the scope of standard diffusion models. The study focuses on refining these equations for enhanced applicability, assimilating them with real-world data, and leading their innovative use in burgeoning technological areas. Through these efforts, the research aims to significantly contribute to scientific and technological progress, offering novel insights and practical solutions in traffic dynamics and broader physical systems.

Lie symmetries, conservation laws, optimal system and exact solutions of (2+1)-dimensional time fractional parabolic equation

J. C. Yu[1], *Y. Q. Feng*[2]

[1]*School of Science, Wuhan University of Science and Technology, Wuhan 430081, Hubei, P. R. China*; yjicheng@126.com

[2]*Hubei Province Key Laboratory of Systems Science in Metallurgical Process, Wuhan 430081, Hubei, P. R. China*; yqfeng6@126.com

In this paper, Lie symmetry analysis method is applied to one type of the (2+1)-dimensional time fractional parabolic equations, which has extensive applications in physics and engineering. All Lie symmetries and the corresponding conserved vectors for the equation are obtained. The one-dimensional optimal system is utilized to reduce the aimed equation with Riemann-Liouville fractional derivative to (1+1)-dimensional fractional partial differential equation with Erdélyi-Kober fractional derivative. Some exact solutions are constructed.

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