Analytical and Numerical Methods in Differential Equations (Yanenko 100 and ANMDE 2021)

Conference Program

and

Book of Abstracts

23 – 27 August 2021

Virtual Host:

Suranaree University of Technology
Nakhon Ratchasima, Thailand

Preface

This year marks the 100th birthday of the well-known Russian mathematician, Academician Nikolai Nikolaevich Yanenko (1921-1984), who made great contributions to the development of both, analytical and numerical methods for solving differential equations, to computational mathematics, and to fluid mechanics. Born in Siberia, Academician Yanenko spent most of his professional life in Siberia, eight years of which as Director of the Institute of Theoretical and Applied Mechanics in Novosibirsk in the then Soviet Union. While he thus is best known in the countries of the former Soviet Union, his work has deeply influenced leading researchers all over the world.

Here at Suranaree University, we are lucky to have with us colleagues and have received numerous visitors who either were Yanenko's students or had the opportunity to learn from him. We are therefore very pleased to host the virtual conference "Analytical and Numerical Methods in Differential Equations", organized jointly with Durban University of Technology in South Africa and the Khristianovich Institute of Theoretical and Applied Mechanics of the Russian Academy of Sciences at Novosibirsk. Due to the current Covid situation, this conference had to take the form of a virtual conference, but we hope to have the opportunity of meeting many of you in person here in Thailand some time in the future.

We are particularly delighted to welcome over 120 researchers from all of the inhabited continents of the world, delivering a total of 80 presentations. Because of this unexpectedly large number of speakers, we have extended the conference by an extra day. We hope that you find this meeting inspiring and can avail of the opportunity to engage in interesting and fruitful discussions with your colleagues and friends.

Our sincere gratitude goes to Suranaree University of Technology and Durban University of Technology for providing financial support. We also wish to thank the latter and the Khristianovich Institute of Theoretical and Applied Mechanics for helping coorganize this conference. This meeting would not have been possible without the scientific contributions of all speakers and the hard work of the members of the School of Mathematics including its students, to whom we extend our deepest gratitude.

Above all, we wish that you enjoy the conference, and extend our warmest greetings to all of you.

The Local Organizing Committee

Conference Program

Day 1 : Monday, 23 August 2021

	Chair: Chaiyasena A. P.
13.00 – 13.05	Opening Address by **Assoc.Prof. Dr. Anan Tongraar** Acting Rector Suranaree University of Technology
13.05 – 13.10	Opening Remarks by **Prof. Dr. Eugene A. Bondar** Deputy Director Khristianovich Institute of Theoretical and Applied Mechanics
13.10 – 13.15	Opening Remarks by **Prof. Dr. Sibusiso Moyo** Deputy Vice Chancellor for Research, Innovation and Engagement Durban University of Technology
13.15 – 13.45	**Fomin V. M.** N.N. Yanenko — Siberian, Soldier and Scientist
13.45 – 14.15	**Il'in, V. P.** (postponed to Friday) The strategies and tactics of an intelligent mathematical modeling
14.15 – 14.40	**Peradzynski Z., Baghaturia G.** Double waves and Yanenko equation
Coffee Break	
	Chair: Pukhnachev V. P.
14.50 – 15.20	**Oberlack M., Hoyas S., Kraheberger S., Alcántara-Álvila F., Laux J., Klingenberg D., Hollmann P., Vallikivi M., Hultmark M., Bellani G., Talamelli A., Zimmerman S., Klewicki J.** Classical and statistical symmetries of turbulence – the basis of turbulent scaling laws of wall-bounded shear flows for arbitrary moments
15.20 – 15.45	**Wacławczyk M, Grebenev V. N., Oberlack M.** Conformal invariance of the 1-point statistics of the zero-isolines of $2d$ scalar fields in inverse turbulent cascades
15.45 – 16.10	**Andreev V. K., Lemeshkova E. N.** On the asymptotic behavior of inverse problems for parabolic equation
16.10 – 16.35	**Grebenev V. N., Demenkov A. G., Chernykh G. G.** Local equilibrium approximation in free turbulent flows: verification through the method of differential constraints
16.35 – 17.00	**Algazin S. D., Selivanov I. A.** (cancelled) About the flutter of an orthotropic plate rectangular in plan
Lunch / Dinner Break	
	Chair: Kaptsov O. V.
18.00 – 18.25	**Manganaro N.** Method of differential constraints for nonlinear wave problems
18.25 – 18.50	**Habibullin I. T.** Generalized invariant manifolds and their applications
18.50 – 19.15	**Dryuma V. S.** The Riemann spaces related to the Navier-Stokes equations
19.15 – 19.40	**Pavlov M. V.** Egorov hydrodynamic type systems
19.40 – 20.10	**Sergyeyev A.** Integrable systems in four independent variables from contact geometry

Coffee Break	
	Chair: Dorodnitsyn V. A.
20.35 – 21.00	**Dyachenko S. A., Zakharov V. E., Dyachenko A. I.** On dynamics of a free boundary in 2D hydrodynamics
21.00 – 21.25	**Anco S. C.** General symmetry multi-reduction method for partial differential equations with conservation laws
21.25 – 21.50	**Olver, P. J.** Higher order symmetries of underdetermined systems of partial differential equations and Noether's second theorem

Day 2 : Tuesday, 24 August 20

	Chair: Pelinovsky E.
13.00 – 13.25	**Grigoriev Yu. N., Meleshko S. V., Siriwat P.** Qualitative properties and invariant solutions of the nonstationary one-dimensional equations of a vibrationally excited gas
13.25 – 13.50	**Stepanyants Y. A.** Scalar description of three-dimensional flows of incompressible fluid
13.50 – 14.15	**Makarenko N. I., Maltseva J. L., Cherevko A. A.** Internal waves in two-layer stratified flows
14.15 – 14.40	**Kukushkin D. E., Makarenko N. I., Shapeev V. P.** Modelling stationary flows in bounded domain
14.40 – 15.05	**Karabut E. A., Zhuravleva E. N.** Using analytical continuation for solving nonlinear free boundary problems
Coffee Break	
	Chair: Makarenko N. I.
15.20 – 15.45	**Gavrilyuk, S., Shyue K.-M.** Singular solutions of the BBM equation: analytical and numerical study
15.45 – 16.10	**Chesnokov A. A., Liapidevskii V. Yu.** Internal solitary waves in a multi-layer stratified fluid: new models and their verification
16.10 – 16.35	**Baenova G. M., Sukhinin S. V., Zhumadillayeva A. K.** Features of the propagation of long Waves in phonon crystals
16.35 – 17.00	**Tkachev D. L.** Lyapunov instability of the polymeric fluid flow in channel (channel walls are perforated)
Lunch / Dinner Break	
	Chair: Muriel C.
18.00 – 18.25	**Halder, A. K., Duba, C. T., Leach, P. G. L.** Symmetries and solutions of the modified nonlinear Schrödinger equation
18.25 – 18.50	**Holba P.** Complete classification of local conservation laws for generalized Kuramoto–Sivashinsky equation
18.50 – 19.15	**Kovtunenko V. A.** Shape gradient method of equilibrium-constrained optimization for semi-linear Stokes–Brinkman–Forchheimer's equation
19.15 – 19.40	**Paliathanasis, A.** Lie symmetries and Bohmian inhomogeneous cosmology
18.40 – 20.05	**Aibinu M. O., Thakur S. C, Moyo S.** On construction of exact solutions to delay reaction-diffusion systems
Coffee Break	

	Chair: Halder, A. K.
20.20 – 20.45	**Bila N.** A study of the Tzitzeica curves equation
20.45 – 21.10	**de la Cruz A., Diaz-Chang T., Liang Ch., Pistora J., Cada M.** (cancelled) (2+1)-dim asymptotic variational theory for light propagating in a non-local nonlinear dissipative medium
21.10 – 21.35	**Diaz-Chang T., de la Cruz A., Liang Ch., Pistora J., Cada M.** (cancelled) Optical Benney-Luke equation
21.35 – 22.00	**Torres R.** Uniform approximation of impulsive differential systems by using a piecewise constant argument

Day 3 : Wednesday, 25 August 2021

	Chair: Chesnokov A. A.
13.00 – 13.25	**Vaneeva O. O.** Normalization property of classes of differential equations and its application in group snalysis
13.25 – 13.50	**Talyshev A. A.** Differential-invariant solutions of the Navier-Stokes equations with respect to one four-dimensional group
13.50 – 14.15	**Dimakis N.** Geodesic equations and nonlocal conservation laws: The exceptional pp-wave case.
14.15 – 14.40	**Siraeva D. T.** Submodels and exact solutions of the gas dynamics equations with state equation of a special form
14.40 – 15.05	**Seesanea A.** Continuous solutions to sublinear elliptic problems
Coffee Break	
	Chair: Paliathanasis, A.
15.20 – 15.45	**Campoamor-Stursberg R.** Branching rules and subduced representations applied to the symmetry breaking of ODEs
15.45 – 16.10	**Halder, A. K., Paliathanasis, A., Almusawa, H., Leach P. G. L.** Solutions of the Boiti-Leon-Manna-Pempinelli equation using symmetries
16.10 – 16.35	**Pinar Z., Orhan Ö.** The symmetries of the fully nonlinear Monge-Ampre equation
16.35 – 17.00	**Naseer S., Raza A., Zaman F. D. and Kara A. H.** Optimal system and conservation laws for the generalized Fisher equation in cylindrical coordinates
Lunch / Dinner Break	
	Chair: Tsarev S. P.
18.00 – 18.25	**Conte R., Grundland A. M.** Reduction of a sine-Gordon system to a sixth order Painlevé equation
18.25 – 18.50	**Kudryashov N. A.** The generalized Hermite polynomials for the Burgers hierarchy and point vortices
18.50 – 19.15	**Askhabov S. N.** `(cancelled)` System Integro-differential equations of the convolution type with an inhomogeneity in the linear part
Coffee Break	

	Chair: Conte R.
19.35 – 20.00	**Halder A. K., Almusawa H., Paliathanasis A., Leach P. G. L.** Comparative analysis of continuous Kadomtsev-Perviashvili-type equations using symmetries
20.00 – 20.25	**Muriel C., Nucci M. C., Romero, J. L.** On chains of differential equations
20.25 – 20.50	**Orhan Ö., Pınar Z.** Exact solutions and linearization of Ermakov-Pinney equation via the nonlocal transformation-symmetry approach
20.50 – 21.15	**Mitsopoulos, A., Tsamparlis, M.** Higher order first integrals of autonomous dynamical systems
Coffee Break	
	Chair: Schulz, E.
21.30 – 21.55	**Cheviakov A., Dutykh D., Assylbekuly A.** Symmetry properties of a family of BBM-type equations
21.55 – 22.20	**Tarayrah M. R., Cheviakov A.** All exact symmetries of higher-order ODEs are stable

Day 4 : Thursday, 26 August 2021

	Chair: Kudryashov N. A.
13.00 – 13.25	**Ulyanov O. N., Rubina L. I.** On some methods of reducing nonlinear partial differential equations to systems of ordinary differential equations
13.25 – 13.50	**Kaptsov O. V., Mirzaokhmedov M. M.** General solutions of some linear equations with variable coefficients
13.50 – 14.15	**Kazakov A. L., Lee M.-G., Lempert A. A.** Exact solutions having diffusion wave type in nonlinear models of thermal conductivity, filtration, and diffusion
14.15 – 14.40	**Isaev V. I., Cherepanov A. N., Shapeev V. P.** Numerical study of heat modes of laser welding of dissimilar metals with an intermediate insert
14.40 – 15.05	**Palymskiy I. B., Palymskiy V. I.** About convection of compressed gas
Coffee Break	
	Chair: Andreev V. K.
15.20 – 15.45	**Sharifullina T. S., Cherevko A. A., Ostapenko V. V.** Numerical modeling of cerebral arterio-venous malformation embolization based on clinical data
15.45 – 16.10	**Tsarev S. P.** Discrete orthogonal polynomials: detection of anomalies of time series and boundary effects of polynomial filters
16.10 – 16.35	**Rogalev A. N.** Symbolic methods for estimating the sets of solutions of ordinary differential equations with perturbations on a finite time interval
16.35 – 17.00	**Kulikov E. K., Makarov A. A.** On approximation functionals to minimal splines
Lunch / Dinner Break	
	Chair: Gavrilyuk, S.
18.00 – 18.25	**Dobrokhotov, S. Yu.** Constructive uniform asymptotics of linear water waves generated by localized sources
18.25 – 18.50	**Bogdanov, A. N.** Dynamics of shock waves in media with longitudinal stratification. The precise evolution
18.50 – 19.15	**Pelinovsky E., Talipova T.** Travelling waves in 1D strongly inhomogeneous media
19.15 – 19.40	**Rozanova O. S.** The Riemann problem for equations of cold plasma
18.40 – 20.05	**Kaptsov E. I., Dorodnitsyn V. A., Meleshko S. V.** Conservativeness of invariant finite-difference schemes
20.05 – 20.30	**Meleshko S. V., Siriwat P.** Complete set of reciprocal transformations of 2D stationary gasdynamics

Coffee Break	
	Chair: Meleshko, S. V.
21.10 – 21.35	**Bobrovskiy V. S., Sinitsyn A. V.** Mathematical modelling of proton migration inside earthquake source by Vlasov-Maxwell system
21.35 – 22.00	**Chernykh G. G., Fomina A. V., Moshkin N. P.** Dynamics of a heated turbulent mixing zone in a linear stratified medium

Day 5 : Friday, 27 August 2021

	Chair: Dobrokhotov, S. Yu.
13.00 – 13.25	**Pukhnachev V. P.** Reading N.N. Yanenko's papers
13.25 – 13.50	**Alekseev G. V., Brizitskii R. V.** Control problems in magnetohydrodynamics for viscous incompressible fluid
13.50 – 14.15	**Vedenyapin V. V., Fimin N. N., Chechetkin V. M., Russkov A. A., Voronina M. Yu.** On the derivation of the equations of electrodynamics and gravitation from the least action principle and the models of the Universe
14.15 – 14.40	**Vasyutkin S. A., Chupakhin A. P.** Differentiation of similar matrices
14.40 – 15.05	**Millionshchikov D.** Liouville equation and combinatorial polynomials
Coffee Break	
	Chair: Moyo S.
15.20 – 15.45	**Bulatov M. V., Solovarova L. S.** On collocation-variation difference schemes for differential-algebraic equations
15.45 – 16.10	**Solovarova L. S., Phuong T. D.** On difference schemes for the second-order differential-algebraic equations
16.10 – 16.35	**Anikin A. Yu., Rykhlov V. V.** Asymptotics for graphene in magnetic field
Coffee Break	
16.50 – 17.15	**Nadjafikhah M., Mahdipour–Sh. A.** Symmetry analysis of the cylindrical Helmholtz equation
17.15 – 17.40	**Nakpim W., Meleshko S. V.** Conservation laws of the relativistic gas dynamics equations in Lagrangian coordinates
17.40 – 18.05	**Il'in, V. P.** The strategies and tactics of an intelligent mathematical modeling
18.05 – 18.30	**Vladimirov, V. A.** Distinguished Limits and Drifts: between Nonuniqueness and Universality
18.30 – 18.35	Closing Remarks by Sergey V. Meleshko

Abstract Collection

Aibinu M. O., Thakur S. C, Moyo S. *On construction of exact solutions to delay reaction-diffusion systems* 1

Alekseev G. V., Brizitskii R. V. *Control problems in magnetohydrodynamics for viscous incompressible fluid* 2

Algazin S. D., Selivanov I. A. *About the flutter of an orthotropic plate rectangular in plan* 3

Anco S. C. *General symmetry multi-reduction method for partial differential equations with conservation laws* 4

Andreev V. K., Lemeshkova E. N. *On the asymptotic behavior of inverse problems for parabolic equation* 5

Anikin A. Yu., Rykhlov V. V. *Asymptotics for graphene in magnetic field* 6

Askhabov S. N. *System Integro-differential equations of the convolution type with an inhomogeneity in the linear part* 7

Baenova G. M., Sukhinin S. V., Zhumadillayeva A. K. *Features of the propagation of long Waves in phonon crystals* 8

Bila N. *A study of the Tzitzeica curves equation* 9

Bobrovskiy V. S., Sinitsyn A. V. *Mathematical modelling of proton migration inside earthquake source by Vlasov-Maxwell system* 10

Bogdanov, A. N. *Dynamics of shock waves in media with longitudinal stratification. The precise evolution* 11

Bulatov M. V., Solovarova L. S. *On collocation-variation difference schemes for differential-algebraic equations* 12

Campoamor-Stursberg R. *Branching rules and subduced representations applied to the symmetry breaking of ODEs* 13

Chernykh G. G., Fomina A. V., Moshkin N. P. *Dynamics of a heated turbulent mixing zone in a linear stratified medium* 14

Chesnokov A. A., Liapidevskii V. Yu. *Internal solitary waves in a multi-layer stratified fluid: new models and their verification* 15

Cheviakov A., Dutykh D., Assylbekuly A. *Symmetry properties of a family of BBM-type equations* 16

Conte R., Grundland A. M. *Reduction of a sine-Gordon system to a sixth order Painlevé equation* 17

de la Cruz A., Diaz-Chang T., Liang Ch., Pistora J., Cada M. *(2+1)-dim asymptotic variational theory for light propagating in a nonlocal nonlinear dissipative medium* 18

Diaz-Chang T., de la Cruz A., Liang Ch., Pistora J., Cada M. *Optical Benney-Luke equation* 19

Dimakis N. *Geodesic equations and nonlocal conservation laws: The exceptional pp-wave case.* 20

Dobrokhotov, S. Yu. *Constructive uniform asymptotics of linear water waves generated by localized sources* 21

Dryuma V. S. *The Riemann spaces related to the Navier-Stokes equations* 22

Dyachenko S. A., Zakharov V. E., Dyachenko A. I. *On dynamics of a free boundary in 2D hydrodynamics* 23

Gavrilyuk, S., Shyue K.-M. *Singular solutions of the BBM equation : analytical and numerical study* 24

Grebenev V. N., Demenkov A. G., Chernykh G. G. *Local equilibrium approximation in free turbulent flows: verification through the method of differential constraints* . . . 25

Grigoriev Yu. N., Meleshko S. V., Siriwat P. *Qualitative properties and invariant solutions of the nonstationary one-dimensional equations of a vibrationally excited gas* 26

Habibullin I. T. *Generalized invariant manifolds and their applications* 27

Halder A. K., Almusawa H., Paliathanasis A., Leach P. G. L. *Comparative analysis of continuous Kadomtsev-Perviashvili-type equations using symmetries* 28

Halder A. K., Paliathanasis A., Almusawa H., Leach P. G. L. *Solutions of the Boiti-Leon-Manna-Pempinelli equation using symmetries* 29

Halder A. K., Duba C. T., Leach P. G. L. *Symmetries and solutions of the modified non-linear Schrödinger equation* 30

Holba P. *Complete classification of local conservation laws for generalized Kuramoto–Sivashinsky equation* 31

Il'in, V. P. *The strategies and tactics of an intelligent mathematical modeling* 32

Isaev V. I., Cherepanov A. N., Shapeev V. P. *Numerical study of heat modes of laser welding of dissimilar metals with an intermediate insert* 33

Kaptsov E. I., Dorodnitsyn V. A., Meleshko S. V. *Conservativeness of invariant finite-difference schemes* 34

Kaptsov O. V., Mirzaokhmedov M. M. *General solutions of some linear equations with variable coefficients* 35

Karabut E. A., Zhuravleva E. N. *Using analytical continuation for solving nonlinear free boundary problems.* 36

Kazakov A. L., Lee M.-G., Lempert A. A. *Exact solutions having diffusion wave type in nonlinear models of thermal conductivity, filtration, and diffusion* 37

Kovtunenko V. A. *Shape gradient method of equilibrium-constrained optimization for semi-linear Stokes–Brinkman–Forchheimer's equation* 38

Kudryashov N. A. *The generalized Hermite polynomials for the Burgers hierarchy and point vortices* 39

Kulikov E. K., Makarov A. A. *On approximation functionals to minimal splines* 40

Kukushkin D. E., Makarenko N. I., Shapeev V. P. *Modelling stationary flows in bounded domain* 41

Makarenko N. I., Maltseva J. L., Cherevko A. A. *Internal waves in two-layer stratified flows* 42

Manganaro N. *Method of differential constraints for nonlinear wave problems* 43

Meleshko S. V., Siriwat P. *Complete set of reciprocal transformations of 2D stationary gasdynamics* 44

Millionshchikov D. *Liouville equation and combinatorial polynomials* 45

Mitsopoulos, A. and Tsamparlis, M. *Higher order first integrals of autonomous dynamical systems* 46

Muriel C., Nucci M. C., Romero, J. L. *On chains of differential equations* 47

Nadjafikhah M., Mahdipour–Sh. A. *Symmetry analysis of the cylindrical Helmholtz equation* 48

Nakpim W., Meleshko S. V. *Conservation laws of the relativistic gas dynamics equations in Lagrangian coordinates* 49

Naseer S., Raza A., Zaman F. D. and Kara A. H. *Optimal system and conservation laws for generalized Fisher equation in cylindrical coordinates* 50

Oberlack M., Hoyas S., Kraheberger S., Alcántara-Álvila F., Laux J., Klingenberg D., Hollmann P., Vallikivi M., Hultmark M., Bellani G., Talamelli A., Zimmerman S., Klewicki J. *Classical and statistical symmetries of turbulence – the basis of turbulent scaling laws of wall-bounded shear flows for arbitrary moments* 51

Olver, P. J. *Higher order symmetries of underdetermined systems of partial differential equations and Noether's second theorem* 52

Orhan Ö., Pınar Z. *Exact solutions and linearization of Ermakov-Pinney equation via the nonlocal transformation-symmetry approach* 53

Paliathanasis, A. *Lie symmetries and Bohmian inhomogeneous cosmology* 54

Palymskiy I. B., Palymskiy V. I. *About convection of compressible gas* 55

Pavlov M. V. *Egorov hydrodynamic type systems* 56

Pelinovsky E., Talipova T. *Travelling waves in 1D strongly inhomogeneous media* 57
Peradzynski Z., Baghaturia G. *Double waves and Yanenko equation* 58
Pinar Z., Orhan Ö. *The symmetries of the fully nonlinear Monge-Ampre equation* 59
Pukhnachev V. P. *Reading N.N. Yanenko's papers* 60
Rogalev A. N. *Symbolic methods for estimating the sets of solutions of ordinary differential equations with perturbations on a finite time interval* 61
Rozanova O. S. *TheRiemann problem for equations of cold plasma* 62
Seesanea A. *Continuous solutions to sublinear elliptic problems* 63
Sergyeyev A. *Integrable systems in four independent variables from contact geometry* . . 64
Sharifullina T. S., Cherevko A. A., Ostapenko V. V. *Numerical modeling of cerebral arterio-venous malformation embolization based on clinical data* 65
Siraeva D. T. *Submodels and exact solutions of the gas dynamics equations with state equation of a special form* 66
Solovarova L. S., Phuong T. D. *On difference schemes for the second-order differential-algebraic equations* 67
Stepanyants Y. A. *Scalar description of three-dimensional flows of incompressible fluid* . 68
Talyshev A. A. *Differential-invariant solutions of the Navier-Stokes equations with respect to one four-dimensional group* 69
Tarayrah M. R. , Cheviakov A. *All exact symmetries of higher-order ODEs are stable* . . 70
Tkachev D. L. *Lyapunov instability of the polymeric fluid flow in channel (channel walls are perforated)* 71
Torres R. *Uniform approximation of impulsive differential systems by using a piecewise constant argument* 72
Tsarev S. P. *Detection of anomalies of time series* 73
Ulyanov O. N., Rubina L. I. *On some methods of reducing nonlinear partial differential equations to systems of ordinary differential equations* 74
Vaneeva O. O. *Normalization property of classes of differential equations and its application in group analysis* 75
Vasyutkin S. A., Chupakhin A. P. *Differentiation of similar matrices* 76
Vedenyapin V. V., Fimin N. N., Chechetkin V. M., Russkov A. A., Voronina M. Yu. *On the derivation of the equations of electrodynamics and gravitation from the least action principle and the models of the Universe* 77
Vladimirov, V. A. *Distinguished limits and drifts: between nonuniqueness and universality* 78
Wacławczyk M, Grebenev V. N., Oberlack M. *Conformal invariance of the* 1*-point statistics of the zero-isolines of* 2d *scalar fields in inverse turbulent cascades* 79
Webb, G. M. *Conservation laws and action principles for CGL plasmas* 80

On Construction of Exact Solutions to Delay Reaction-diffusion Systems

M. O. Aibinu[1], *S. C. Thakur*[2], *S. Moyo*[3]

[1]*Institute for Systems Science & KZN e-Skills CoLab, Durban University of Technology, Durban 4000, South Africa*;

[1]*DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), Johannesburg, South Africa*; moaibinu@yahoo.com / mathewa@dut.ac.za

[2]*KZN CoLab, Durban University of Technology, Durban 4000, South Africa*; thakur@dut.ac.za

[3]*Institute for Systems Science & Office of the DVC Research, Innovation & Engagement Milena Court, Durban University of Technology, Durban 4000, South Africa*; dvcrie@dut.ac.za

Several physical phenomena are governed by mathematical models and can be structured by the Reaction-Diffusion (RD) systems. RD systems are essential for the description of dynamical processes in chemistry, biology, geology, physics (neutron diffusion theory) and ecology, to mention just a few. In general, RD systems have the form

$$\partial_t \mathbf{u} = \mathbf{D}\nabla^2\mathbf{u} + \mathbf{R}(\mathbf{u}), \tag{1}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{r}, \mathbf{t})$ represents the unknown vector function, $\mathbf{D}$ is a diagonal matrix of diffusion coefficients, ∇^2 is the Laplace operator which acts on the vector $\mathbf{u}$ componentwise and $\mathbf{R}$ accounts for all local reactions. Construction of solutions of RD equations of the form (1) with a single unknown function has been studied in [1, 2, 3, 4]. Predator-prey, competition of species for a common food source and symbiotic relationship between two species are common examples of models which cannot be described by RD equations with a single unknown function. This study presents a number of new exact solutions to nonlinear RD systems of the form (1) with delay and which have multiple unknown functions. The solutions to RD systems with delay which are presented in this study are suitable to formulate test problems intended to evaluate the accuracy of numerical methods for solving nonlinear delay PDEs.

REFERENCES

1. Polyanin A. D. and Sorokin V. G., *A method for constructing exact solutions of nonlinear delay PDEs*, J. Math. Anal. Appl. 494 (2021) 124619.
2. Polyanin A. D. and Sorokin V. G., *New exact solutions of nonlinear wave type PDEs with delay*, Applied Mathematics Letters 108 (2020) 106512.
3. Meleshko S. V. and Moyo S., *On the complete group classification of the reaction-diffusion equation with a delay*, J. Math. Anal. Appl. 338 (2008) 448-466.
4. Pandey P., Kumar S., Gómez-Aguilar J. F. and Baleanu D., *An efficient technique for solving the space-time fractional reaction-diffusion equation in porous media*, Chinese Journal of Physics 68 (2020) 483-492.

Control Problems in Magnetohydrodynamics for Viscous Incompressible Fluid

G. V. Alekseev, R. V. Brizitskii

Institute of Applied Mathematics FEB RAS, Vladivostok, Russia; alekseev@iam.dvo.ru

During recent years control theory for hydrodynamic, thermal and electromagnetic fields in liquid media has been intensively developed. One of the aims of the theory is to establish the most effective mechanism for controlling physical fields in continuous media. The mathematical description of this type of problems includes three components: the purpose, the control mechanisms used to achieve the desired purpose, and the constraints such that the state and the controls of the model under consideration should satisfy them. The role of constraints is usually played by hydrodynamic, magnetohydrodynamic (MHD), electromagnetic and some other equations together with boundary and initial conditions, while the desired purpose is achieved by minimizing a certain cost functional.

The problems of controlling MHD-flows of electrically and heat-conducting fluid historically have first arisen in metallurgy and foundry during the development of optimal technologies for contactless electromagnetic stirring of molten metals and in the nuclear industry in the creation of effective liquid-metal cooling systems for nuclear power units. Later, the necessity for solving control problems was brought about by the problems arising in the creation of plants for the industrial growth of crystals by the methods of melting and dissolution and the development of new submarine engines [1].

This paper presents some results obtained by the authors in a rigorous study of control problems for two models of magnetohydrodynamics of viscous fluid. The first model consists of the Navier-Stokes equations for the dynamics of viscous incompressible fluid and Maxwell's equations without exterior currents for electromagnetic field, interconnected through the Lorentz force and the generalized Ohm's law for moving fluid. The second model is obtained by adding the convection-diffusion equation for the temperature of medium and other additional terms, which take into account the influence of thermal effects of the movement of fluid, to the first model. The results obtained by the authors are partially presented in [1].

The authors gratefully acknowledge financial support by the Ministry of Science and Higher Education of the Russian Federation (project no. 075-02-2020-1482-1, additional agreement of 21.04.2020)

REFERENCES

1. Alekseev G. V. and Brizitskii R. V., *Boundary control problems for the stationary magnetic hydrodynamic equations in domain with non-ideal boundary*, J. Dynamical and Control Systems. **26**(4), 641–661 (2020).

About the Flutter of an Orthotropic Plate Rectangular in Plan

S. D. Algazin[1], *I. A. Selivanov*[2]

[1]*Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russian Federation*; algazinsd@mail.ru
[2]*Lomonosov Moscow State University, Moscow, Russian Federation*; shertors@gmail.com

The authors' previous paper [1] presented the results of solving the problem of free vibrations of an orthotropic clamped rectangular plate. Below we consider the flutter of a pinched orthotropic plate, streamlined, on one side, by an airflow. The mathematical model of plate flutter constructed by A.A. Ilyushin and I.A. Kiyko [2] is accepted. The effective algorithm for solving the problem was developed by the first author, and I.A. Kiyko [3]. The basis of the technique is the construction of a discrete bi-harmonic operator [4]. The software package is arranged in such a way that it is possible to find the critical flutter velocity and build the corresponding eigenform in an arbitrary direction of the airflow velocity vector. As standard, the critical flutter velocity is searched for on two grids. Previously, this problem in a slightly different formulation was considered in [5] in the direction of the velocity vector of airflow only along the x-axis. The flutter of an isotropic rectangular plate was previously considered in [6]. In this paper, these results are generalized to a rectangular orthotropic plate.

Consider a rectangular plate that occupies a region in the xy plane $S : \{-1 \le x \le 1,\ -b \le y \le b\}$:

$$L\phi + \beta\vec{V} grad\phi = \lambda\phi, \quad \rho h\omega^2 + \beta\omega + \lambda = 0, \quad \beta = kp_0/c_0, \quad \vec{V} = (V_x, V_y)\,. \tag{1}$$

$$x, y \in \Gamma, \quad \phi = 0, \quad M(\phi) = 0. \tag{2}$$

$$L\phi = D_x\frac{\partial^4\phi}{\partial x^4} + 2D_{xy}\frac{\partial^4\phi}{\partial x^2\partial y^2} + D_y\frac{\partial^4\phi}{\partial y^4}.$$

Let us apply an interpolation formula for the function $\phi = \phi(x, y)$ in (1),(2) in the rectangle that satisfies the marginal clamping conditions:

$$\begin{aligned}
&\phi(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{i0}(z)L_{j0}(x)\phi(x_j, y_i)\,,\\
&y = bz, z \in [-1, 1], \quad x \in [-1, 1];\\
&L_{j0}(x) = \tfrac{l(x)}{l'(x_j)(x-x_j)}, \quad l(x) = (x^2-1)^2 T_n(x), \quad T_n(x) = \cos n \arccos x;\\
&x_j = \cos\theta_j, \quad \theta_j = (2j-1)\pi/(2n), \quad j = 1, 2, ..., n;\\
&M_{i0}(z) = \tfrac{M(z)}{M'(z_i)(z-z_i)}, \quad M(z) = (z^2-1)^2 T_m(z);\\
&z_i = \cos\theta_i, \quad \theta_i = (2i-m)\pi/(2m), \quad i = 1, 2, ...m.
\end{aligned}$$

In order to obtain a discrete operator L matrix, it is required to apply this operator to the interpolation formula. As a result, we obtain an asymmetric matrix H of size $N \times N$, $N = mn$. Let us first number the nodes in the rectangle (x_j, y_i) by X and then by Y, i.e. from top to bottom, from right to left. As a result, we obtain that Lw is approximately replaced by the relation Hw, where ϕ– is the vector of function values $\phi = \phi(x, y)$ in the nodes of the grid.

REFERENCES

1. Algazin S. D. and Selivanov I. A., *Fluctuations of the orthotropic clamped rectangular plates*, Scientific Bulletin of The State Scientific Research Institute of Civil Aviation, **31**, 140–148 (2020).
2. Ilyushin A. A. and Kiyko I. A., *New formulation of the Flutter Problem of a Hollow Shell*, J. Appl. Math. Mech., **58**(3), 167–171 (1994).
3. Algazin S. D. and Kiyko I. A., *Numerical-analytical study of the flutter of an arbitrarily shaped plate in the plan*, J. Appl. Math. Mech., **61**(1), 171–174 (1997).
4. Algazin S. D., *Numerical algorithms of classical matophysics. II. Spectral Problems for the biharmonic Equations* , Ishlinsky Institute for Problems in Mechanics RAS, **678** (2001).
5. Papkov S. O., *Flutter of a clamped orthotropic rectangular plate*, Computational Continuum Mechanics, **10**(4), 361–374 (2017).
6. Algazin S. D. and Kiyko I. A., *Numerical study of the flutter of a rectangular plate*, Journal of Applied Mechanics and Technical Physics, **44**(4), 35–42 (2003).

General Symmetry Multi-reduction Method for Partial Differential Equations with Conservation Laws

S. C. Anco

Department of Mathematics and Statistics, Brock University, St Catharines, Canada; sanco@brocku.ca

A powerful application of symmetries is finding symmetry-invariant solutions of nonlinear differential equations. These solutions satisfy a reduced differential equation with one fewer independent variable. It is well known that a double reduction occurs whenever the starting nonlinear differential equation possesses a conservation law that is invariant with respect to the symmetry.

Recent work has developed a broad generalization of the double-reduction method by considering the space of invariant conservation laws with respect to a given symmetry group. The generalization is able to reduce a nonlinear partial differential equation (PDE) in n variables to an ODE with $m - n + 2$ first integrals where m is the dimension of the space of invariant conservation laws.

In this talk, a summary of the general multi-reduction method will be presented, with applications to obtaining invariant solutions of physically interesting PDEs.

REFERENCES

1. Sjoberg A., *Double reduction of PDEs from the association of symmetries with conservation laws with applications*, Appl. Math. Comput., **184**, 608616 (2007).
2. Anco S. C., *Symmetry properties of conservation laws*, Int. J. Mod. Phys. B, **30**, 1640004 (2016).
3. Anco S. C. and Kara A., *Symmetry invariance of conservation laws*, Euro. J. Appl. Math., **29**(1), 78–117 (2018).
4. Anco S. C. and Gandarias, M. L., *Symmetry multi-reduction method for partial differential equations with conservation laws*, Commun. Nonlin. Sci. Numer. Simul., **91**, 105349 (2020).

On the Asymptotic Behavior of Inverse Problems for Parabolic Equation

V. K. Andreev[1], *E. N. Lemeshkova*[2]

[1]*Institute of Computational Modelling SB RAS, Krasnoyarsk, Russia*; andr@icm.krasn.ru
[2]*Institute of Computational Modelling SB RAS, Krasnoyarsk, Russia*;
elena_cher@icm.krasn.ru

We study two inverse initial-boundary value problems for a linear parabolic equation. These equations arise in mathematical modeling of the viscous heat-conducting fluid motion with two or one free boundaries. The unknown function of time enters the right-hand side of the equation additively and is found from the additional condition of integral overdetermination. For both problems, a priori estimates of solutions in the uniform metric are obtained. Stationary solutions are found. Sufficient conditions for the input data, under which the solutions with increasing time tend to the stationary regime according to the exponential law, are established.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement No. 075-02-2020-1631).

Constructive Semi-classical Asymptotic Formulas for Quasimodes of Dirac Operator Describing Graphene in Magnetic Field

A. Yu. Anikin, V. V. Rykhlov

Institute for Problems in Mechanichs of the Russian Academy of Sciences, Moscow, Russia; anikin83@inbox.ru

We propose constructive semi-classical asymptotics for the eigenfunctions of the Dirac operator describing graphene in a constant magnetic field. Two cases are considered: (a) a strong magnetic field, and (b) a radially symmetric electric field with low mass. The problem is reduced by standard semi-classical methods to a pencil of magnetic Schrödinger operators with a correction. In both cases, the classical system defined by the main symbol turns out to be integrable, but the correction destroys the integrability. In case (a), where the correction removes the frequency degeneracy (resonance), using the averaging method, we reduce the problem to an integrable system not only in the leading approximation, but also with the correction taken into account. The tori of the resulting system generate a series of asymptotic eigenfunctions of the original operator. In case (b), the system defined by the main symbol is nondegenerate. Fixing an invariant torus with Diophantine frequencies for this system and finding a solution of the transport equation for it, we obtain a series of asymptotic eigenfunctions that are in one-to-one correspondence with tori that satisfy the Bohr-Sommerfeld rule and lie in a small neighborhood of the chosen Diophantine torus. In both cases, the construction of the asymptotics of the eigenfunctions is based on the global representation in terms of the Airy function and its derivative for the Maslov's canonical operator on a two-dimensional torus projected onto the configuration space into an annular domain with two simple caustics. We also give some numerical examples that illustrate that the obtained formulas are efficient.

The work was supported by the Russian Science Foundation (project No. 16-11-10282).

System Integro-Differential Equations of the Convolution Type with an Inhomogeneity in the Linear Part

S. N. Askhabov

Chechen State Pedagogical University, Chechen State University, Grozny, Russia;
askhabov@yandex.ru

Questions are considered concerning the existence, uniqueness, search and properties of solutions to a system of nonlinear integro-differential equations of the form::

$$u_i^{\alpha}(x) = \sum_{j=1}^{n} \int_0^x k_{ij}(x-t)\cdot u_j'(t)\,dt + f_i(x), \quad \alpha > 1\,, \quad x > 0\,, \quad i = \overline{1,n}\,, \tag{1}$$

where $k(x) = \{k_{ij}(x)\}_{i,j=1}^{n}$ $f(x) = \{f_i(x)\}_{i=1}^{n}$ satisfy on $[0,\infty)$ the conditions:

$$k_{ij} \in C^2[0,\infty),\ k_{ij}'(x) \text{ is not decreasing on } [0,\infty),\ k_{ij}(0) = 0 \text{ and } k_{ij}'(0) = p_{ij} > 0. \tag{2}$$

$$f_i \in C^1[0,\infty), f_i(x) \text{ is not decreasing on } [0,\infty) \text{ and } f_i(0) = 0. \tag{3}$$

In connection with applications in hydrodynamics, models of population genetics, etc. (for more details see [1]), solutions to system (1) are sought in the cone

$$Q_{0,n}^1 = \{u : u = \{u_i\}_{i=1}^n, u_i \in C[0,\infty) \cap C^1(0,\infty),\ u_i(0) = 0\ \ u_i(x) > 0\ \ x > 0\}.$$

A priori estimates for the solution of system (1) are obtained, on the basis of which the global theorem on the existence and uniqueness of the solution is proved by the method of weighted metrics and it is shown that it can be found by the method of successive approximations. In particular, the following are true

Lemma 1. *Let conditions (2) and (3) be satisfied. If $u \in Q_{0,n}^1$ is a solution to system (1), then $F_n(x) \le u_i(x) \le G_n(x)$ $x \in [0,\infty)$ $i = \overline{1,n}$, where*

$$F_n(x) \equiv \left[\frac{(\alpha-1)\,n\,p}{\alpha}\right]^{1/(\alpha-1)} x^{1/(\alpha-1)}, \quad G_n(x) \equiv \left[n \sum_{i,j=1}^{n} k_{ij}(x) + \left(\sum_{i=1}^{n} f_i(x)\right)^{(\alpha-1)/\alpha}\right]^{1/(\alpha-1)},$$

$p = \min\limits_{1\le i,j\le n} p_{ij}$.

Theorem 1. *If conditions (2), (3) are satisfied, $\sup\limits_{0<x\le b} \left(\sum\limits_{i=1}^{n} f_i(x)\right)^{(\alpha-1)/\alpha} / x < \infty$, where $b > 0$ is any number, and $\max\limits_{1\le i\le n} \sum\limits_{j=1}^{n} k_{ij}'(\eta_{ij}) < \alpha\cdot p\cdot n$ for some $\eta_{ij} > 0$, then the system of equations (1) has a unique solution in $Q_{0,n}^1$. This solution can be found by the method of successive approximations.*

Under more general assumptions with respect to the kernel $k(x)$, it is proved that in the case $0 < \alpha < 1$ system (1) with $f = 0$ can have only a trivial solution $u = 0$.

The work was carried out as part of the implementation of the state assignment for the project "Nonlinear singular integro-differential equations and boundary value problems" in accordance with the Agreement of December 29, 2020 N 075-03-2021-071.

REFERENCES

1. Askhabov S. N., *Integro-differential equation of the convolution type with a power nonlinearity and an inhomogeneity in the linear part*, Differential Equations, **56**(6), 775–784 (2020).

Features of the Propagation of Long Waves in Phonon Crystals, the Influence of the Concentration and Polydispersity of the components

G. M. Baenova[1], *S. V. Sukhinin*[2], *A. K. Zhumadillayeva*[3]

[1]*L.N. Gumilyov ENU, Nur-Sultan, Kazakhstan*; baenova_gm@enu.kz
[2]*Institute of Hydrodynamics SB RAS, Novosibirsk, Russia*; sukhinin@hydro.nsc.ru
[3]*L.N. Gumilyov ENU, Nur-Sultan, Kazakhstan*; zhumadillayeva_ak@enu.kz

The relevance of the research is determined by the need to model wave propagation in heterogeneous media. These studies are necessary for the tasks of flaw detection, the development of non-destructive testing methods, geophysics, engineering geophysics and other methods of acoustic research. Direct studies of wave propagation in such media are impossible, therefore, methods of averaging the main characteristics of a heterogeneous medium are almost always used. Modeling of heterogeneous media using phonon crystals is essentially one of these methods. A fundamental cell characteristic of a heterogeneous medium is determined, which contains 2 connected regions filled with one of the components "1" and "2", the sizes of these regions are determined by volume concentrations. Such a medium can be considered a monodisperse phonon crystal, if the fundamental cell contains more than 1 region with component "1"and component "2", then it is polydisperse. It should be noted that the waves in phonon crystals are dispersing [1, 4]. This paper presents the results of numerical and analytical studies of the effect of polydispersity of two-component phonon crystals on the propagation of long waves in the first transmission band at the same concentrations of the two components.

1. For the first time, a significant effect of polydispersity on the dispersion relations, phase and group velocities for long dispersing waves in a phonon crystal was discovered.
2. The dispersion of the phonon crystal components can be close to the formation and decay of the fundamental cell into smaller ones by 2-3 times. In this case, the phase and group velocities also change in the corresponding number of times.

The results of numerical and analytical studies of wave propagation in mono-and polydisperse structures for known components of phonon crystals, water - air, porous concrete, brick, and others are presented. For example, a fundamental cell was taken as part of a medium consisting of two permeable media in the form of a chain of air and water bubbles. The solution and characteristics of the oscillations are obtained in comparison with a monodisperse medium in which all the components of the media become close in size to each other.

REFERENCES

1. Brillouin L. and Parodi M., *Wave propagation in periodic structures*, Izd. inostr. lit., Moscow (1959).
2. Sukhinin S. V., *Wave propagation and resonant phenomena in inhomogeneous media*, J. Appl. Mech. Tech. Phys. 2001, **42**(3), pp. 411-419.
3. Sukhinin S. V. and Kondratenko D. A., *Wave propagation in strongly non homogeneous wave guide.*, Proc. Int. Conf. on MMET (Kharkov), 1998, pp. 2-5.
4. Saimanova Z. B., Sukhinin S. V. and Zhumadillayeva A. K., *Wave transmission and reflection from the boundary of phononic crystal homogeneous medium*, Eurasian Journal of Mathematical and Computer Applications. 2020, **8**(1), pp. 62-75.

A Study of the Tzitzeica Curves Equation

N. Bila

Department of Mathematics and Computer Science, Fayetteville State University, Fayetteville, North Carolina, United States; nbila@uncfsu.edu

The Tzitzeica curve equation is an intriguing nonlinear ordinary differential equation arising in differential geometry that is satisfied by a space curve for which the ratio of its torsion and the distance from the origin to its osculating plane at an arbitrary point of the curve is constant. The class of curves has been introduced by the Romanian geometer Gheorghe Tzitzeica in 1911 in his study on affine invariants. Nowadays, there are known only a few examples of Tzitzeica curves defined explicitly in terms of the elementary functions. Interestingly, although the Tzitzeica curves have occurred occasionally in the mathematics literature, the ordinary differential equation defining these curves has not been studied extensively so far, maybe due to the fact that Tziteica curves are defined by a nonlinear ordinary differential equation whose unknowns are the curves defining functions. The aim of this talk is to present several techniques for finding Tzitzeica curves along with symmetry reductions associated with their defining equation. A side condition involving the Wronskian of the curves defining functions is considered. It is shown that the Tzitzeica curve equation can be reduced to an auxiliary third order linear homogeneous ordinary differential equation with constant coefficients for the defining functions of the curve and a linear equation for the equation's constant. An interesting connection between Tzitzeica curves and generalized hypergeometric functions is also introduced. Additionally, a systematic method for determining Tzitzeica curves is proposed, and new solutions are presented.

Mathematical Modelling of Proton Migration Inside Earthquake Source by Vlasov-Maxwell System

V. S. Bobrovskiy[1] *and A. V. Sinitsyn*[2]

[1] *Cosmetecor UK, England; Department of Management and Engineering, Politecnico di Milano, Italy*; vadokoem@gmail.com

[2] *Departamento de Matemáticas, Universidad Nacional de Colombia, Bogota, Colombia*; avsinitsyn@yahoo.com

We consider the Vlasov-Maxwell (VM) system with external magnetic field to model protonic hydrogen migration for travelling wave solutions to ground motion problems. Firstly, we prove existence of quasi-stationary solutions of VM system in a bounded domain for nonlinear nonlocal elliptic system via the sub-super solution method. The next part provides the existence theorem for the Poisson external boundary value problem. The magnetic field is found as a coupling of solutions of two boundary value problems: the first one for the potential U(x,t) of self-consistent electromagnetic field and the second one is external boundary value problem for potential of magnetic field. Further, a numerical illustration of the results is considered. The fast finite element method is used to solve the three-dimensional nonlinear hyperbolic equation in the bounded domains supplemented by the transport equation through inflow condition. Model reports new scientific result: first imaging of proton migration giving a unique source location. The model can be used to produce accurate strong earthquakes final source predictions ahead of time and to assess the uncertainty regarding the factors controlling the ground motion. Decision makers, either public and private, can use this information to make decisions, against a risk metric, on the mobilization of a wide range of capacities to mitigate earthquake consequences.

Dynamics of Shock Waves in Media with Longitudinal Stratification. The Precise Evolution

A. N. Bogdanov

Research Institute of Mechanics, Moscow State University, Moscow;
bogdanov@imec.msu.ru

The study of the dynamics of shock waves (SW) in inhomogeneous gaseous media is one of the topical problems in connection with applications [1]. Problem statements of this kind are a natural development of classical studies on the dynamics of shock waves in a homogeneous medium [2]. Chiznell is considered the first in this direction [3]. The complexity of the study of these problems in the general case initially determined the way to search for their particular solutions under certain kind of conditions. For some types of density stratification ahead of the SW front, self-similar solutions were obtained: Sakurai [4] investigated such solutions in the case of a power-law density change; for an exponential decrease in density, self-similar solutions were constructed by Hayes [5]. Plane, axisymmetric, and spherical shock waves in a gas of variable density varying according to a power law in the direction of wave motion were investigated in self-similar regimes by F.L. Chernousko [6]. The propagation of hydrocarbons in a gaseous medium with an exponential density distribution was investigated by A.S. Kompaneets [7]. A very original mathematical approach to the problem of propagation of a one-dimensional shock wave in a quiescent polytropic gas with a given, one-dimensional, pressure distribution was presented by L.V. Ovsyannikov [8]. An overview of the achievements made by the end of the 1970s and the presentation of his own original results in this area is devoted to Ch. 8 of the monograph by J. Whitham [9]. Although Whitham's results were quite close to [4] and [5], the originality of the method for obtaining the dependence of the perturbation rate on the parameters of the medium left questions (acknowledged by the author [9]) about the validity of the results obtained. Numerical calculations of the motion of shock waves in an inhomogeneous medium were carried out in [10]. Cases of the passage of a hydrocarbon layer of constant increased or decreased density and temperature at constant pressure, as well as a different molecular weight and adiabatic exponent, were considered. The results were presented in the form of graphs, analytical dependences for them were not displayed. An analytical approach to the study of shock waves in inhomogeneous media was proposed [11] by the author of this work, the results obtained in this and other ways in their development are presented in the report.

Asymptotic methods are very flexible, acting decisively, one can get an answer of a rather complex problem - this rule was perceived by the author through O.S. Ryzhov.

REFERENCES

1. Glass I. I., *Shock Waves and Man*, Mir, Moscow, 191pp (1977).
2. Sedov L. I., *Similarity and Dimensional Methods in Mechanics*, Nauka, Moscow, 448pp (1981).
3. Chisnell R. F., *The normal motion of a shock wave through a nonuniform one-dimensional medium*, Proc. Roy. Soc. A, **232**, 350–370 (1955).
4. Sakurai A., *On the problem of a shock wave arriving at the edge of a gas*, Comm. Pure Appl. Math., **13**, 353–370 (1960).
5. Hayes W. D., *Self-similar strong shocks in an exponential medium*, J. Fluid Mech., **32**, 305–315 (1968); (Russian translation in: Coll. Mechanics, **112**(6), 51–61 (1968))
6. Chernousko, F. L., *Converging shock waves in a gas of variable density*, Prikl., **24**(5), 885–896 (1960).
7. Zeldovich Ya. B. and Raizer Yu. P., *Physics of Shock Waves and High-temperature Hydrodynamic Phenomena*, Nauka, Moscow, 688pp (1966).
8. Ovsyannikov L. V., *Approximate method of recalculation of the law of propagation of shock waves*, PMTF, No. 1, 55–57 (1972).
9. Whitham G. B., *Linear and Nonlinear Waves*, John Wiley & Sons. (1974), 635pp. (Russian translation: Mir, Moscow, 622pp (1977))
10. Levin V. A. and Laskovy M. V., *Propagation of blast waves in an inhomogeneous medium*, Report of the Research Institute of Mechanics, Moscow State University, No. 3776, 70pp (1989).
11. Bogdanov A. N., *Dynamics of shock waves in media with longitudinal stratification*, Dokl. RAN.T., **491**, 5–6 (2020). (English translation: Dokl. Phys., **65**, 83–84 (2020))

On Collocation-Variation Difference Schemes for Differential-Algebraic Equations

M. V. Bulatov[1], *L. S. Solovarova*[2]

[1]*Matrosov Institute for System Dynamics and Control Theory of SB RAS, Irkutsk Russia*; mvbul@icc.ru
[2]*Matrosov Institute for System Dynamics and Control Theory of SB RAS, Irkutsk Russia*; soleilu@mail.ru

Consider the system of linear differential equations

$$A(t)x'(t) + B(t)x(t) = f(t),\ t \in [0, 1], \tag{1}$$

$$x(0) = x_0, \tag{2}$$

where $A(t)$, $B(t)$ are $(n \times n)$-matrices, $f(t)$ and $x(t)$ are the given and the unknown n-dimensional vector-functions, respectively.

If

$$det A(t) \equiv 0, \tag{3}$$

then systems of the form (1),(2) are called differential-algebraic equations (DAEs). It is assumed that input data are smooth enough for further reasoning and the solution satisfies the initial condition (2).

The difficulties of numerical solution of the DAE and ordinary differential equations are discussed. Collocation-variation difference schemes is proposed for solving problem (1),(2) with condition (3). Their construction is based on an idea from [1],[2]. The analysis of the particular cases of the schemes and the numerical calculations of the test examples are given.

The authors gratefully acknowledge financial support by RFBR grants 20-51-S52003, 20-51-54003, 18-29-10019.

REFERENCES

1. Bulatov M. V., Gorbunov V. K., Martynenko Yu. V. and Cong N. D., *Variational approaches to numerical solution of differential algebraic equations*, Computational Technologies, **5**, 3–13 (2010).
2. Bulatov M. V. and Solovarova L. S., *Collocation-variation difference schemes with several collocation points for differential-algebraic equations*, Appl. Numer. Math., **149**, 153–163 (2020).

Branching Rules and Subduced Representations Applied to the Symmetry Breaking of Systems of Differential Equations

R. Campoamor-Stursberg

Instituto de Matemática Interdisciplinar, Universidad Complutense, E-28040 Madrid, Spain; rutwig@ucm.es

Realizations of Lie algebras in terms of vector fields in connection with the representation theory and the branching rule problem associated to embeddings of simple Lie algebras are considered. This allows to determine if a subalgebra in a given realization corresponds to an irreducible embedding, as well as to determine multiplicities in the branching rules. The invariants of the realizations associated to such embeddings determine second-order (non-conservative Lagrangian) dynamical systems invariant by a certain symmetry group G (of Lie point and/or Noether symmetries). The analysis of the branching rules and the corresponding invariants provides an effective tool to decide whether the symmetry can be broken to a given subgroup $K \subset G$, eventually leading to an algorithmic construction of dynamical systems with exact symmetry group K and given conservation laws.

The author gratefully acknowledges financial support by the research grants MTM2016-79422-P (AEI/FEDER, EU) and PID2019-106802GB-I00/AEI/10.13039/501100011033 (AEI/ FEDER, UE).

REFERENCES

1. Patera J. and Sankoff D., *Tables of Branching Rules for Representations of Simple Lie Algebras*, Presses de lUniversité de Montréal, Montréal (1973).
2. Campoamor-Stursberg R., *Reduction by invariants and projection of linear representations of Lie algebras applied to the construction of nonlinear realizations*, J. Math. Phys., **59**(3), 033502 (2018).
3. Ibragimov N. H. (Ed), *CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 2. Applications in Engineering and Physical Sciences*, CRC Press, Boca Raton, FL, (1994).

Dynamics of a Heated Turbulent Mixing Zone in a Linear Stratified Medium

G. G. Chernykh[1,2], *A. V. Fomina*[3], *N. P. Moshkin*[2,4]

[1]*Federal Research Center for Information and Computational Technologies, Novosibirsk, Russia*; chergg19@gmail.com
[2]*Novosibirsk State University, Novosibirsk, Russia*;
[3]*Novokuznetsk Institute (Branch) of Kemerovo State University, Novokuznetsk, Russia*; fav@rdtc.ru
[4]*Lavrentyev Institute of Hydrodynamics SB RAS, Novosibirsk, Russia*; nikolay.moshkin@gmail.com

Evolution of localized regions of turbulized fluid (turbulent spots) has a considerable effect on the formation of fine microstructure of hydrophysical fields in the ocean [1]. A brief overview of research on the dynamics of turbulent spots in a stably stratified fluid at rest can be found in [2]. The evolution of laminar spots of mixed liquid of non-zero buoyancy (thermals) plays an important role in the formation of fine structures of ocean water, creation of clouds, and many other natural phenomena [3].

In the present work, the averaged Naiver-Stokes equations with the Oberbeck-Boussinesq approximation are used to construct a numerical model of the dynamics of a flat heated turbulized region of non-zero buoyancy in a linearly stratified medium. Unknown values of the dissipation rate and Reynolds stresses are found via numerical integration of the differential equations. The turbulent fluxes and density fluctuation variance are found from the locally equilibrium algebraic relations. The algorithm to the problem solution uses finite-difference methods based on the explicit splitting in the physical processes and spatial variables method with weighted approximation of convective terms. At each time step the equations of turbulence characteristics are solved with application of implicit splitting method on spatial variables [4].

It is shown that a weakly heated laminar localized region of mixed fluid generates internal waves of a significantly greater amplitude in comparison with a spot of non-zero buoyancy. Presence of non-zero buoyancy leads to essential increase in the geometrical dimensions of the turbulent spot and generation of internal waves of greater amplitude in comparison with an evolution of turbulent spot of zero buoyancy. The recent work is a continuation and development of research [2].

REFERENCES

1. Monin A. S. and Yaglom A. M., *Statistical Fluid Mechanics. Vol. 1. Mechanics of Turbulence*, Dover Books on Physics. (2007), 784pp.
2. Chernykh G. G., Fomina A. V. and Moshkin N. P., *Numerical Simulation of Dynamics of Weakly Heated Turbulent Mixing Zone in Linearly Stratified Medium*, Journal of Engineering Thermophysics, **29**(4), 674–685 (2020).
3. Antropov I. V. and Kronrod V. A., *On Evolution of a Thermal in a Stratified Medium in Dependence on Initial Conditions*, Izv. AN SSSR, FAO, **25** (12), 1261–1266 (1989).
4. Yanenko N. N., *The Method of Fractional Steps; the Solution of Problems of Mathematical Physics in Several Variables*, Berlin, New York, Springer-Verlag, (1971), 160pp.

Internal Solitary Waves in a Multilayer Stratified Fluid: New Models and Their Verification

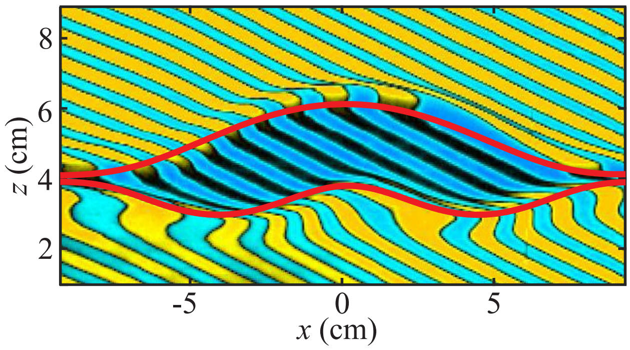
A. A. Chesnokov[1], *V. Yu. Liapidevskii*[2]

[1]*Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia*; chesnokov@hydro.nsc.ru

[2]*Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia*; liapid@hydro.nsc.ru

We derive a new family of mathematical models describing the propagation of internal waves in a stratified shallow water with a non-hydrostatic pressure distribution in the Boussinesq approximation. The construction of the models is based on the use of additional instantaneous variables. This allows one to reduce the dispersive multilayer GreenNaghdi-type model to a first-order system of evolution equations. The advantage of the proposed models is the simplicity of numerical implementation and realization of non-reflecting boundary conditions.

We consider three-layer flows over an uneven bottom with the additional assumption of hydrostatic pressure in the intermediate layer [1]. The hyperbolicity conditions of the obtained equations for three-layer flows are formulated, and solutions in the class of travelling waves are studied. Numerical simulation of the propagation and interaction of symmetric and non-symmetric soliton-like waves is performed. Numerical calculations of the generation and propagation of internal solitary waves are carried out and their comparison with experimental data (Deepwell et al, 2019; Liapidevskii and Gavrilov, 2018) is given. In particular, mode-2 non-symmetric internal solitary wave described by the obtained model is shown in the figure. Bold solid curves correspond to the interfaces; coloured picture presents snapshot of experiment Liapidevskii and Gavrilov, 2018; blue colour inside of the wave shows the initially coloured fluid trapped by the wave.



More general models, including an arbitrary number of intermediate hydrostatic layers, are also derived and used to describe the propagation of large-amplitude near-bottom and/or near-surface internal waves [2]. Stationary solutions of the governing equations are studied and conditions for the formation of mode-1 internal solitary waves are formulated. The results of numerical modelling based on these equations are verified by comparison with field observations (Preusse et al, 2012; Lien et al, 2014; Liapidevskii et al, 2017; Kukarin et al, 2019).

This work was supported by the Russian Science Foundation (project 20-11-20189).

REFERENCES

1. Chesnokov A. A. and Liapidevskii V.Yu., *Hyperbolic model of internal solitary waves in a three-layer stratified fluid*, Europ. Phys. J. Plus, **135**(7), 590 (2020).
2. Liapidevskii V.Yu., Chesnokov A. A. and Ermishina V. E., *Quasilinear equations for the dynamics of internal waves in multilayer shallow water*, J. Appl. Mech. Tech. Phys., **62**(4), (forthcoming) (2021).

Symmetry Properties of a Family of Benjamin-Bona-Mahony-type Equations

A. Cheviakov[1], *D. Dutykh*[2], *A. Assylbekuly*[3]

[1]*Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada*; cheviakov@math.usask.ca
[2]*Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France*; Denys.Dutykh@univ-smb.fr
[3]*Department of Mathematics, Novosibirsk State University, Novosibirsk, Russia*; Aidar.Asylbekuly@kaznu.kz

The Benjamin-Bona-Mahony (BBM) equation, originally derived by Peregrine [1] is given by

$$u_t + u_x + uu_x - u_{xxt} = 0\,. \tag{BBM}$$

It possesses Hamiltonian and Lagrangian descriptions, exact solitary wave-type solutions, and has a bounded dispersion relation. An important drawback of (BBM) as a physical model is the absence of the Galilean invariance. A modified version of the PDE (BBM) recovering the Galilean invariance and having the same order of approximation was proposed in Ref. [2]

$$u_t + u_x + uu_x - u_{xxt} - uu_{xxx} = 0\,. \tag{iBBM}$$

Unlike the original BBM model, the Galilei-invariant PDE (iBBM) lacks the conservation of energy. The remedy to that is a further modification with the same order of approximation,

$$u_t + u_x + uu_x - u_{xxt} - uu_{xxx} - 2u_x u_{xx} = 0\,. \tag{eBBM}$$

which is both Galilei-invariant and energy-preserving. We further introduce a one-parameter family of equations

$$u_t + u_x + uu_x - u_{xxt} - A\,(uu_{xxx} + 2u_x u_{xx}) = 0\,, \tag{A}$$

that include both the (BBM) when $A = 0$, and the (eBBM) when $A = 1$. All PDEs (A) share the Hamiltonian and Lagrangian structures. We perform a comprehensive study local and nonlocal symmetries and conservation laws of the family (A). In particular, cases with additional local conservation laws are the (eBBM) ($A = 1$) and the additional case $A = 1/3$:

$$u_t + u_x + uu_x - u_{xxt} - \frac{1}{3}\,uu_{xxx} - \frac{2}{3}\,u_x u_{xx} = 0 \tag{eBBM$_{1/3}$}$$

possessing higher-order symmetries and a large number of additional conservation laws. We show that (eBBM$_{1/3}$) is related to the Camassa-Holm equation

$$u_t + 3uu_x - 2u_x u_{xx} - u_{xxt} - uu_{xxx} = 0\,.$$

by a simple point transformation. We also consider traveling wave reductions and perform numerical investigations of bump evolution, energy conservation, and solitary wave interaction of the BBM-type models (BBM), (iBBM), (eBBM), and the integrable PDE (eBBM$_{1/3}$).

A.C. is grateful to NSERC of CANADA for research support through the Discovery grant RGPIN-2019-05570. D.D. acknowledges support from the Fédération de Recherche en Mathématiques Auvergne-Rhône-Alpes (FR 3490). The work of D.D. has also been supported by the French National Research Agency, through Investments for Future Program (ref. ANR-18-EURE-0016–Solar Academy). The research of A.A. was supported by Kazakhstan Ministry of Education and Science (AP08053154).

REFERENCES

1. Peregrine D. H., *Calculations of the development of an undular bore*, J. Fluid Mech., **25**(2), 321–330 (1966).
2. Duran A., Dutykh, D., and Mitsotakis, D H., *On the Galilean invariance of some nonlinear dispersive wave equations*, Stud. Appl. Math., **131**(4), 359–388 (2013).
3. Cheviakov A., Dutykh D., and Assylbekuly A., *On Galilean invariant and energy preserving BBM-type equations*, Symmetry, **13**(5), 878 (2021).

Reduction of a Sine-Gordon System to a Sixth Order Painlevé Equation

R. Conte[1,2], *A. M. Grundland*[3]

[1] *Université Paris-Saclay, ENS Paris-Saclay, CNRS Centre Borelli, F-91190 Gif-sur-Yvette, France*; Robert.Conte@cea.fr

[2] *Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong*;

[3] *Département de Mathématiques et d'Informatique, Université du Québec à Trois-Rivières, Québec, G9A 5H7, Canada*; Grundlan@CRM.UMontreal.CA

We establish all the reductions of the system of two coupled 2+1-dimensional sine-Gordon equations introduced by Konopelchenko and Rogers to ordinary differential equations (ODE). There is only one such reduction, to an ODE of Chazy, algebraic transform of the sixth Painlevé equation. Various degeneracies also lead to the fifth, third and second Painlevé equations and to elliptic functions.

(2+1)-dim Asymptotic Variational Theory for Light Propagating in a Nonlocal Nonlinear Dissipative Medium

A. de la Cruz[1], *T. Diaz-Chang*[2], *Ch. Liang*[3], *J. Pistora*[4], *M. Cada*[1]

[1]*Electrical and Computer Engineering Dpt, Dalhousie University, Halifax, Canada*; artorix@dal.ca
[2]*Instituto de Ciencias Fisicas y Matematicas, Universidad Austral de Chile, Chile*;
[3]*College of Physics, Soochow University, Suzhou 215006, China*;
[4]*Nanotechnology Centre and IT4I, VSB - Technical University of Ostrava, 708 33 Ostrava-Poruba, Czech Republic*;

We propose and demonstrate analytically, within the framework of a hydrodynamic model, a novel and simpler variational approach to study the asymptotic behavior of a continuous wave (cw) laser beam propagating in a weakly absorbing defocusing nonlinear nonlocal media. The Kadomtsev-Petviashvili (KP) type equation is obtained. For the first time, to the best of our knowledge, the variational multiscale asymptotics method is used to describe nonlinear open systems. The starting point in the analysis is the light propagation in a weakly nonlocal nonlinear defocusing medium described by [1] normalized NLSE with a dissipative term and a diffusion-like equation for the response of the nonlocal medium

$$i\,\epsilon\,\frac{\partial\Psi}{\partial z}+\frac{\epsilon^2}{2}\nabla^2\,\Psi-\Theta\,\Psi=-i\,\epsilon\,\frac{\alpha}{2}\Psi\quad\text{and}\quad-\,\sigma^2\,\nabla^2\,\Theta+\Theta=|\Psi|^2,\tag{1}$$

where $\nabla^2=\partial_x^2+\partial_y^2$. The z and $\mathbf{r}=(x,y)$ are the spatial evolutionary variable and the transverse coordinates, respectively. Ψ is the complex electric field envelop peak intensity, Θ is a real function that denotes the nonlinear nonlocal change of the refractive index depending on the intensity. α is the intensity loss rate and $\epsilon \ll 1$ is a small quantity that deal with the weakly diffracting regime. The parameter σ is a spatial scale (setting the diffusion length) that measures the degree of nonlocality. A hydrodynamic model with the help of the Mandelung transformation $\Psi(z,\mathbf{r})=\rho^{1/2}(z,\mathbf{r})\,\exp[\,i\,h(z,\mathbf{r})\,]$. Both functions Ψ and Θ are assumed to be non-zero at the boundaries (infinities). Using $\Psi(z,\mathbf{r})=\psi_b(z)\,\psi(z,\mathbf{r})$ and $\Theta(z,\mathbf{r})=\theta_b(z)\,\varphi(z,\mathbf{r})$ in the above system of equations, the background equations $\psi_b(z)$ and $\theta_b(z)$ are to be determined as well. We propose that above system of equations can be derived from the appropriate Lagrangian density

$$L=\rho\left[\frac{(\nabla h)^2}{2}+\frac{\partial h}{\partial z}+\varphi-1\right]-\frac{1}{2}\left[\varphi^2+(\sigma\nabla\varphi)^2-1\right]+\frac{\left(\nabla\sqrt{\rho}\right)^2}{2}.$$

By means of the stretched variables $\xi=\epsilon^{1/2}\,(x-z)$, $\eta=\epsilon\,y$, $\tau=\epsilon^{3/2}\,z$, it is possible to write the Lagrangian as

$$L=\epsilon\,L^{(1)}+\epsilon^2\,L^{(2)}+\epsilon^3\,L^{(3)}+\mathcal{O}(\epsilon^4).$$

Introducing the variable transformations $\tau\to-(8\gamma)\,\tau$, $\eta\to(\sqrt{3|\gamma|/2})\,\eta$ and $u=-(\gamma/2)\,U$, we arrive to a KP equation [2]

$$\partial_\xi\left(U_\tau+6\,U_\xi^2+U_{\xi\xi\xi}+3\,\varsigma^2\,U_{\eta\eta}\right)=0.$$

where $\varsigma^2=-\mathrm{sgn}\gamma$.

Artorix de la Cruz thanks the financial support from Killam Trust Predoctoral and Nova Scotia Research scholarships.

REFERENCES

1. Ghofraniha N, Conti C., Ruocco G. and Trillo S., Phys. Rev. Lett. **99**, 043903 (2007).
2. Horikis T. P. and Frantzeskakis D. J., Phys. Rev. Lett. **118**, 243903 (2017).

Optical Benney-Luke Equation

T. Diaz-Chang[1], *A. de la Cruz*[2], *Ch. Liang*[3], *J. Pistora*[4], *M. Cada*[2]

[1]*Instituto de Ciencias Fisicas y Matematicas, Universidad Austral de Chile, Chile*;
tamara.diaz@uach.cl
[2]*Electrical and Computer Engineering Dpt, Dalhousie University, Halifax, Canada*;
[3]*College of Physics, Soochow University, Suzhou 215006, China*;
[4]*Nanotechnology Centre and IT4I, VSB - Technical University of Ostrava, 708 33 Ostrava-Poruba, Czech Republic*;

We analyze, within the framework of an Optical Benney-Luke equation (OBLE), the light propagation in a nonlocal nonlinear defocusing media. The exact solitary wave (SW) profiles, for the light intensity and its phase chirp, have been obtained analytically in terms of the optical surface tension, which depends on the degree of nonlocality. The solutions dynamics have been demonstrated numerically. Our results show that the OBLE satisfies the reported "homeomorphism" between optics and shallow-water waves and gives an insight into the nonlocal nonlinear Schrödinger equation (NLSE) evolution in the intermediate asymptotics regime. The starting point in the analysis is the light propagation in a weakly nonlocal nonlinear defocusing medium described by [1] normalized NLSE with a dissipative term and a diffusion-like equation for the response of the nonlocal medium

$$i\,\epsilon\,\frac{\partial\Psi}{\partial z}+\frac{\epsilon^2}{2}\,\frac{\partial^2\,\Psi}{\partial x^2}-\Theta\,\Psi=0 \quad \text{and} \quad -\,\sigma^2\,\nabla^2\,\Theta+\Theta=|\Psi|^2, \tag{1}$$

The z and x are the spatial evolutionary variable and the transverse coordinates, respectively. Ψ is the complex electric field envelop peak intensity, Θ is a real function that denotes the nonlinear nonlocal change of the refractive index depending on the intensity. The $\epsilon \ll 1$ is a small quantity that deal with the weakly diffracting regime. The parameter σ is a spatial scale (setting the diffusion length) that measures the degree of nonlocality. The solutions can be proposed in the form $\psi=\psi_0\sqrt{\rho}\exp(-i|\psi_0|^2 z+i\,\epsilon^{1/2}\Phi)$. A scaled Benney-Luke equation [2] is obtained as

$$\frac{\partial^2\Phi}{\partial Z^2}-C^2\,\frac{\partial^2\Phi}{\partial X^2}+\epsilon\left[\frac{\sigma}{4}\,\frac{\partial^4\Phi}{\partial X^4}+\frac{1}{2}\,\frac{\partial}{\partial Z}\left(\frac{\partial\Phi}{\partial X}\right)^2+\frac{\partial}{\partial X}\left(\frac{\partial\Phi}{\partial Z}\frac{\partial\Phi}{\partial X}\right)\right]=0, \tag{2}$$

Let us propose traveling wave solutions in the form $\Phi(X,Z)=V(X-Z)\equiv V(\eta)$. Integrating once and changing the variable as $\varphi=dV/d\eta$, we obtain the following relation

Integrating (2) one obtains the phase Φ associated with this solution, which in terms of the original (dimensionless) coordinates, x and z reads:

$$\Phi(x,z)=\sqrt{\frac{-\,\gamma\,H}{\epsilon}}\;\;\tanh\left[\frac{1}{W}\;x-v_g\;z\right]\;, \tag{3}$$

where $v_g=\sqrt{-(\epsilon\,H)/\gamma}$ and $W=-(\gamma/H)\sqrt{-H/\gamma}, H=(1-C^2)/\epsilon$.

As a function of the original (dimensionless) z and x, one may write down an approximate [up to order $\mathcal{O}(\epsilon)$] solution for the macroscopic wavefunction $\psi\sim\psi_0\,(\rho_0+\epsilon\,\rho_1)^{1/2}$ and $n=n_0+\epsilon\,n_1$ as follows:

$$\psi=\psi_0\,\sqrt{1-\frac{\epsilon^{1/2}}{|\psi_0|^2}\,\frac{\partial\Phi}{\partial z}}\;\;\exp(-i|\psi_0|^2z+i\epsilon^{1/2}\Phi), \quad \text{and} \quad n=|\psi_0|^2-\epsilon^{1/2}\,\frac{\partial\Phi}{\partial z}\;, \tag{4}$$

REFERENCES

1. Ghofraniha N., Conti C., Ruocco G. and Trillo S., Phys. Rev. Lett. **99**, 043903 (2007).
2. Horikis T. P. and Frantzeskakis D. J., Phys. Rev. Lett. **118**, 243903 (2017).

Geodesic Equations and Nonlocal Conservation Laws: The Exceptional pp-Wave Case

N. Dimakis

Center for Theoretical Physics, College of Physics, Sichuan University, Chengdu 610064, China; nsdimakis@scu.edu.cn

In this work we review nonlocal conservation laws for geodesic systems of massive particles and their connection to the existence of proper conformal Killing vectors [1]. In the case where the metric is that of a pp-wave geometry, we demonstrate that the aforementioned conserved quantities acquire, equivalent on the mass shell, local expressions [2]. The latter are generated by disformal transformations of the metric. We observe how the resulting generators can be obtained as "point symmetries" by slightly modifying the Noether symmetry approach. Furthermore, we study their relation to actual Noether symmetries, of higher order, which produce rational in the momenta integrals of motion for the geodesic problem.

REFERENCES

1. Dimakis N., Terzis P. A. and Christodoulakis T., *Integrability of geodesic motions in curved manifolds through nonlocal conserved charges*, Phys. Rev. D, **99**, 104061 (2019).
2. Elbistan M., Dimakis N., Andrzejewski K., Horvathy P. A. and Kosínski P., *Conformal symmetries and integrals of the motion in pp waves with external electromagnetic fields*, Annals Phys., **418**, 168180 (2020).

Constructive Uniform Asymptotics of Linear Water Waves Generated by Localized Sources

S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. A. Tolchennikov

Ishlinsky Institute for Problem in Mechanics, Russian Academy of Sciences, Moscow 119526, Russia; s.dobrokhotov@gmail.com

We discuss an elementary constructive method for constructing effective uniform asymptotics in wide neighborhoods of simple standard and non-standard caustics for linear differential and pseudodifferential equations. The term simple caustic means that it is an $(n-1)$- D surface in the n-D physical space of the problem under consideration We illustrate the method by the example of the Cauchy-Poisson problem for the linear water waves over an uneven bottom with localized initial data. Non-standard caustics in this case is the leading front edge.

The Riemann Spaces Related to the Navier-Stokes Equations

V. S. Dryuma

Institute of Mathematics and Computer Science, Kishinev, Moldova;
valdryum@gmail.com

For solving system of the Navier-Stokes equations $\frac{\partial}{\partial t}\vec{V}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\mu\Delta\vec{V}+\vec{\nabla}P(\vec{x},t),\ \vec{\nabla}\cdot\vec{V}=0$, where $\vec{V}(\vec{x},t)$ -is the fluid velocity, $P(\vec{x},t)$- is the pressure and μ- is the viscosity of liquid, was used their representation in form of the laws of conservations:

$$\frac{\partial}{\partial y}H(\vec{x},t)-\frac{\partial}{\partial x}E(\vec{x},t)=0,\frac{\partial}{\partial z}H(\vec{x},t)-\frac{\partial}{\partial x}B(\vec{x},t)=0,\frac{\partial}{\partial z}E(\vec{x},t)-\frac{\partial}{\partial y}B(\vec{x},t)=0$$

This allow us to introduce six-dimension Riemann space, equipped by the metric $ds^2=-2\,B(\vec{x},t)dt\,dv+2\,E(\vec{x},t)dt\,dw+2\,H(\vec{x},t)dv\,dw$ $-2(\int\frac{\partial}{\partial y}H(\vec{x},t)dz)dw^2+dt\,dx+dv\,dy+dw\,dz$ and to use it for construct an examples of solutions of the NS-equations. In particular case when the components of metric are of the form $B(\vec{x},t)=\frac{\partial^2}{\partial z^2}Q(\vec{x},t)$,$H(\vec{x},t)=\frac{\partial^2}{\partial x\partial z}Q(\vec{x},t)E(\vec{x},t)=\frac{\partial^2}{\partial y\partial z}Q(\vec{x},t)$ solutions of the NS-equations are expressed throw the function $Q(\vec{x},t)=\frac{\partial P}{\partial y}$, that is solution of the Monge-Ampere equation (MA):

$$2\,Q_{xyzz}^2-2\,Q_{yyxz}Q_{xzzz}-2\,Q_{xxyz}Q_{yzzz}+Q_{xxzz}Q_{zzyy}+Q_{xxyy}Q_{zzzz}=0. \tag{1}$$

Theorem 1. *The equation (1) determines function pressure $P(\vec{x},t)=\int(Q(\vec{x},t)dt)$ of flow and in particular cases admit reduction to the second order ODE of the form $y''+a_1(x,y)y'^3+3a_2(x,y)y'^2+3a_3(x,y)y'+a_4(x,y)=0$, where $a_i=a_i(x,y)$. This type of ODE's meet in theory of nonlinear dynamical systems $\dot{x}=F_i(\vec{x})$ with polynomial right parts which have as the limit cycles and also strange attractors in their space of states.* As an example, we give the equation

$$\frac{d^2}{dx^2}y(x)-3\,\frac{\left(\frac{d}{dx}y(x)\right)^2}{y(x)}-2\,\frac{(k-6\,x+3)\left(\frac{d}{dx}y(x)\right)y(x)}{k+3}-12\,\frac{(y(x))^3\,x^2}{k+2}+\frac{(4\,k+12)\,(y(x))^3\,x}{k+2}+$$
$$+\frac{(-k-2)\,(y(x))^3}{k+2}-6\,\frac{(y(x))^4\,x^4}{(k+3)\,(k+2)}-2\,\frac{(-6-2\,k)\,(y(x))^4\,x^3}{(k+3)\,(k+2)}-2\,\frac{\left(k^2+3\,k+3\right)(y(x))^4\,x^2}{(k+3)\,(k+2)}=0,$$

which has a form similar to the equation $y''-3\,\frac{y'^2}{y}+\left(\alpha\,y-x^{-1}\right)y'+\epsilon\,xy^4+\frac{\delta}{x}y^2-\gamma\,y^3-\beta\,x^3y^4-\beta\,x^2y^3=0$, which is equivalent to the Lorenz-system $\dot{y}=rx-y-xz,\ \dot{z}=xy-bz,\ \dot{x}=\sigma(y-x)$, where: $\alpha\sigma=1,\ \beta\sigma^2=1,\ \gamma\sigma^2=b(\sigma+1),\ \delta\sigma=(\sigma+1),\epsilon\sigma^2=b(r-1)$.

A more general approach to study properties of the NS-equations is connected by using the 14-dimension space with local coordinates $x,y,z,t,u,v,w,p,\xi,\eta,\chi,\rho,q,\delta$.

Theorem 2. The metric: $ds^2=2\,dxdu+2\,dydv+2\,dzdw+(-W(\vec{x},t)w-V(\vec{x},t)v-U(\vec{x},t)u)\,dt^2+\left(-U(\vec{x},t)p-u\,(U(\vec{x},t))^2-uP(\vec{x},t)+w\mu\,\frac{\partial}{\partial z}U(\vec{x},t)-wU(\vec{x},t)W(\vec{x},t)\right)d\eta^2+$
$+\left(v\mu\,\frac{\partial}{\partial y}U(\vec{x},t)-vU(\vec{x},t)V(\vec{x},t)+u\mu\,\frac{\partial}{\partial x}U(\vec{x},t)\right)d\eta^2+2\,\eta d\xi+2\,d\rho d\chi+2\,dmdn+$
$+\left(-V(\vec{x},t)p-vP(\vec{x},t)-v\,(\vec{x},t))^2-V(\vec{x},t)W(\vec{x},t)w+v\mu\,\frac{\partial}{\partial y}V(\vec{x},t)-uU(\vec{x},t)V(\vec{x},t)\right)d\rho^2+$
$+\left(u\mu\,\frac{\partial}{\partial x}V(\vec{x},t)\right)d\rho^2+\left(-uU(\vec{x},t)W(\vec{x},t)-w\,(W(\vec{x},t))^2-wP(\vec{x},t)+w\mu\,\frac{\partial}{\partial z}W(\vec{x},t)\right)dm^2+$
$+\left(v\mu\,\frac{\partial}{\partial y}W(\vec{x},t)-vV(\vec{x},t)W(\vec{x},t)+u\mu\,\frac{\partial}{\partial x}W(\vec{x},t)-W(\vec{x},t)p\right)dm^2$, is the Ricci-flat on solutions of the NS-equations.

This metric belongs to the class of partially-projective spaces and their Cartan-invariants, $K=R(^i\ a\ j\ b)R(^j\ c\ i\ d)A^aA^bA^cA^d$, constructed on the basis of the Riemann curvature of the space are used then in theory of the NS-equations.

REFERENCES

1. Dryuma V. S., *The Riemann and Einstein geometries in the theory of ODE's, their appications and all that*, New Trends in Integrability and Partial Sovaability., **Kluwer academic Publisher**(A.B. Shabat at al.(eds.), 115–156 (2004).
2. Dryuma V. S., *On spaces related to the Navier-Stokes equations*, Buletinul Academiei de Stiinte a Republicii Moldova., **Matematica, 3(64)**, 107–110 (2010).

On Dynamics of a Free Boundary in 2D Hydrodynamics

S. A. Dyachenko[1], *V. E. Zakharov*[2,3,4], *A. I. Dyachenko*[2]

[1] *Department of Mathematics, University at Buffalo,14260-2900, USA*;
[2] *Landau Institute for Theoretical Physics, 142432, Chernogolovka, Russia*;
[3] *Skolkovo Institute of Science and Technology, Moscow, Russia*;
[4] *Department of Mathematics, University of Arizona, Tucson, AZ, 857201, USA*;
alexd@itp.ac.ru

A potential flow of an ideal incompressible fluid with a free surface and infinite depth is considered in the 2D geometry. The fluid dynamics can be fully characterized by the motion of the complex singularities in the analytical continuation of both the conformal mapping and the complex velocity. One of the possible singularities is the square root cut. We derived the exact equations describing the evolution of this cut along with complex velocity given on its sides. The equations show that in general case surface remains smooth at all times. Analytical results are supported by numerical simulations.

Supported by RSF grant No. 19-72-30028

Singular Solutions of the BBM Equation : Analytical and Numerical Study

S. Gavrilyuk[1], *K.-M. Shyue* [2]

[1]*Aix-Marseille Univ, France*; sergey.gavrilyuk@univ-amu.fr
[2]*Institute of Applied Mathematics, NTU, Taiwan*; shyue@ntu.edu.tw

We show that the Benjamin-Bona-Mahony (BBM) equation admits stable travelling wave solutions representing a sharp transition front linking a constant state with a periodic wave train. The constant state is determined by the parameters of the periodic wave train : the wave length, amplitude and phase velocity, and satisfies both the Rankine-Hugoniot conditions for the corresponding Whitham modulation system and generalized Rankine-Hugoniot conditions for the exact BBM equation. Such stable shock-like travelling structures exist if the phase velocity of the periodic wave train is not less than the solution average value. To validate the accuracy of the numerical method, we derive the (singular) solitary limit of the Whitham system for the BBM equation and compare the corresponding numerical and analytical solutions. We find good agreement between analytical results and numerical solutions.

REFERENCES

1. Gavrilyuk, S. and Shyue K.-M. *Singular solutions of the BBM equation : analytical and numerical study*, https://hal.archives-ouvertes.fr/hal-03220465.
2. Gavrilyuk, S. and Shyue K.-M. *Hyperbolic approximation of the BBM equation*, https://hal.archives-ouvertes.fr/hal-03255398.

Local Equilibrium Approximation in Free Turbulent Flows: Verification Through the Method of Differential Constraints

V. N. Grebenev[1], *A. G. Demenkov*[2], *G. G. Chernykh*[1]

[1]*Federal Research Center for Information and Computational Technologies, Novosibirsk, Russia*; vngrebenevp@gmail.com

[2]*Kutateladze Institute of Thermophysics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, Russia* ; demenkov@itp.nsc.ru

We present a version of the results obtained in Grebenev et al [ZAMM Zeitschrift fur Angewandte Mathematik und Mechanik. - 2021. - Art.e202000095] wherein the closure formula, that is, the local equilibrium approximation of second-order moments for modeling free turbulent flows was justified by the method of differential constrains. The proposed analysis provides us a point of view from the modern theory of symmetry analysis on the closure problem in turbulence. Specifically, closure relationships in the physical space are interpreted as the (differential) equations of invariant sets (manifolds) in a phase-space. We demonstrate how this concept can be applied for verification of the local equilibrium approximations (LEA) of second-order moments. With this, we obtain the equivalence of LEA and vanishing the Poisson bracket for the defect of the longitudinal velocity component and of the turbulent energy. Numerical experiments carried out in a far turbulent wake confirm this conclusion.

Qualitative Properties and Invariant Solutions of the Nonstationary One-dimensional Equations of a Vibrationally Excited Gas

Yu. N. Grigoriev[1], *S. V. Meleshko*[2], *P. Siriwat*[3]

[1]*Institute of Computational Technology, Novosibirsk, Russia*; grigor@ict.nsc.ru
[2]*School of Mathematics, Institute of Science, Suranaree University of Technology, Nakhon Ratchasima, Thailand*; sergey@math.sut.ac.th
[3]*School of Science, Mae Fah Luang University, Chiang Rai, Thailand*; piyanuch.sir@mfu.ac.th

Mathematical properties of the system of nonstationary one-dimensional equations describing flows of an inviscid vibrationally excited gas are investigated. Conservation laws in divergent differential and integral forms were found for the cases of cylindrical and spherical symmetries. Under condition that the relaxation time τ is constant, the complete admitted Lie algebras (groups) were calculated for the system. Optimal systems of the one-dimensional subalgebras are constructed, corresponding representations of invariant solutions and reduced equations for them are obtained and some particular solutions were found in an explicit form. It is shown that, in contrast to a similar system for an ideal gas, admitted Lie algebras do not contain a generator for simultaneous scaling the independent variables, which is associated with known self-similar solutions of the problems of strong shock waves. In order to obtain the generalizations of these solutions that are interesting from a physical point of view for the case of an vibrationally excited gas, a modification of the Landau - Teller equation for the vibrational temperature of the gas is proposed. This made it possible to include the necessary generator in the Lie algebra admitted by the modified equations, and derive a system in self-similar variables. As an example, the problem of a strong point explosion is solved on its basis, for which the known effect of lagging the growth of the vibrational temperature from the static (translational) one behind the shock wave front is obtained.

Generalized Invariant Manifolds and Their Applications

I. T. Habibullin

Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Sciences, Ufa, Russia; habibullinismagil@gmail.com

The method of differential constraints is a widely known method for constructing explicit solutions for differential equations [1]. In the talk we will discuss a generalization of the method when the constraint is assigned not to the given equation but to its formal linearization (Fréchet derivative). Such kind generalization of the invariant manifold (differential constraint) finds important application in the soliton theory, since it provides an effective tool for constructing Lax pairs and recursion operators, it allows also to derive Dubrovin type equations which are fundamental objects in the finite-gap integration method (see [2], [3]).

REFERENCES

1. Sidorov A. F., Shapeev V. P. and Yanenko N. N., Method of Differential Constraints and its Applications in Gas Dynamics, Nauka (1984).
2. Habibullin I. T., Khakimova A. R. and Poptsova M. N., *On a method for constructing the Lax pairs for nonlinear integrable equations*, Journal of Physics A: Mathematical and Theoretical, **49**, 035202, 35pp (2016).
3. Habibullin I. T. and Khakimova A. R., *Invariant manifolds and separation of the variables for integrable chains*, Journal of Physics A: Mathematical and Theoretical, **53**(38), 385202, 17pp (2020).

Comparative Analysis of Continuous Kadomtsev-Perviashvili-Type Equations Using Symmetries

A. K. Halder[1], *H. Almusawa*[2], *A. Paliathanasis*[34], *P. G. L. Leach*[35]

[1]*Department of Mathematics, Pondicherry University, Puducherry 605014, India*; amlanhalder1@gmail.com

[2]*Department of Mathematics, College of Sciences, Jazan University, Jazan 45142, Saudi Arabia*; haalmusawa@jazanu.edu.sa

[3]*Institute for Systems Science, Durban University of Technology, PO Box 1334, Durban 4000, South Africa*;

[4]*Instituto de Ciencias FÍsicas y MatemÁticas, Universidad Austral de Chile, Valdivia, Chile*; anpaliat@phys.uoa.gr

[5]*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa*; leachp@ukzn.ac.za

The continuous Kadomtsev-Petviashvili-type (KP) equations are focussed in the form of AKP, BKP, and CKP equations with respect to its symmetries from point and nonlocal perspective. The subsequent reductions for each form and the comparative analysis is presented broadly. For the the first form, AKP, the symmetries are generally infinite-dimensional in nature whereas for the other forms the symmetries are slightly different in nature. A subsequent analysis of the reduced equations are done using singularity analysis to study the integrability of the equations with the presence of series solutions. Certain solutions are also presented using the approach of conservation laws.

Solutions of the Boiti-Leon-Manna-Pempinelli Equation Using Symmetries

A. K. Halder[1], *A. Paliathanasis*[2,3], *H. Almusawa*[4], *P. G. L. Leach*[2,5]

[1]*Department of Mathematics, Pondicherry University, Puducherry - 605014, India*; amlanhalder1@gmail.com

[2]*Institute for Systems Science, Durban University of Technology, PO Box 1334, Durban 4000, Republic of South Africa*;

[3]*Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile*; anpaliat@phys.uoa.gr

[4]*Department of Mathematics, College of Sciences, Jazan University, Jazan 45142, Saudi Arabia*; almusawah@mymail.vcu.edu

[5]*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa*; leachp@ukzn.ac.za

Point symmetries of the Boiti-Leon-Manna-Pempinelli equation [1] in $(1+3)-$ dimension are computed and the corresponding reductions are studied thoroughly. Under a special case the travelling-wave reduction is considered and the subsequent integrability using the method of Singularity analysis is presented. The equation is also studied to compute its nonlocal symmetries using the standard approach and certain recently developed algorithms [2] and a comparison with its local counterpart is mentioned. Singularity analysis of the parent PDE is also mentioned elaborately to emphasize the integrabitility of the equation in a general sense.

AKH is grateful to NBHM Post-Doctoral Fellowship, Department of Atomic Energy (DAE), Government of India, Award No: 0204/3/2021/R&D-II/7242 for financial support. PGLL acknowledges the support of the National Research Foundation of South Africa, the University of KwaZulu-Natal and the Durban University of Technology.

REFERENCES

1. Xu G. Q., *Painlevé analysis, lump-kink solutions and localized excitation solutions for the* $(3+1)$-*dimensional Boiti-Leon-Manna-Pempinelli equation*, Applied Mathematics Letters, **97**, 81-87 (2019).
2. Temuer C., Tong L. and Bluman G., *Some connections between classical and nonclassical symmetries of a partial differential equation and their applications*, Mathematics, **8(4)**, 524 (2020).

Symmetries and Solutions of the Modified Nonlinear Schrödinger Equation

A. K. Halder[1], *C. T. Duba*[2], *P. G. L. Leach*[3]

[1]*Department of Mathematics, Pondicherry University, Puducherry - 605014, India*; amlanhalder1@gmail.com

[2]*School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa*; thama.duba1@wits.ac.za

[3]*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa and Institute for Systems Science, Durban University of Technology, Durban, South Africa*; leachp@ukzn.ac.za

The focal part of this work is to study the possible reductions, solutions in closed-forms of the modified nonlinear Schrödinger equation. Inspired by the work of Dysthe[1, 2, 3] the analysis of the modified version is conducted using point and nonlocal symmetries. The equation reduces to a coupled ordinary differential equation using the translation symmetries and subsequently further to lower-order ODE which assists in further exploration. The Singularity analysis method is also employed separately on those reduced highly nonlinear equations which are lacking any possible point symmetries. A gradual connection with the nonlocal symmetries is presented to further the study of new reductions and solutions.

AKH is grateful to NBHM Post-Doctoral Fellowship, Department of Atomic Energy (DAE), Government of India, Award No: 0204/3/2021/R&D-II/7242 for financial support. PGLL acknowledges the support of the National Research Foundation of South Africa, the University of KwaZulu-Natal and the Durban University of Technology.

REFERENCES

1. Dysthe K. B., *Note on a modification to the nonlinear Schrödinger equation for application to deep water waves*, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, **369**(1736), 105-114 (1979).
2. Cousins W. and Sapsis T. P., *Unsteady evolution of localized unidirectional deep-water wave groups*, Physical Review E, **91**(6), 063204 (2015).
3. Peregrine D. H., *Water waves, nonlinear Schrödinger equations and their solutions*, The ANZIAM Journal, **25** (1), 16-43 (1983).

Complete Classification of Local Conservation Laws for Generalized Kuramoto–Sivashinsky Equation

P. Holba

Mathematical Institute, Silesian University in Opava, Opava, Czech Republic;
pavel.holba@math.slu.cz

In this talk we consider the generalized Kuramoto–Sivashinsky equation, a PDE for $u = u(x_1, \ldots, x_n, t)$ of the form

$$u_t = a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u), \tag{1}$$

where a is a nonzero constant, b, f, g are arbitrary smooth functions of the dependent variable u, $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator, $|\nabla u|^2 = \sum_{i=1}^n (\partial u/\partial x_i)^2$, and n is an arbitrary natural number.

Using an approach close to that from [1] we give a complete classification of all cases when (1) admits nontrivial local conservation laws of any order and give an explicit form of these conservation laws modulo trivial ones.

Equation (1) is a natural generalization of the well-known Kuramoto–Sivashinsky equation arising in a variety of physical and chemical contexts, describing inter alia flame propagation, reaction-diffusion systems and unstable drift waves in plasmas, see e.g. [2] and references therein. The equation in question is recovered from (1) for $a = -1$, $b = -1$, $f = -1/2$ and $g = 0$.

The author gratefully acknowledges financial support from Specific Research Grant SGS/13/2020 of the Silesian University in Opava.

REFERENCES

1. Holba P. *Nonexistence of local conservation laws for generalized Swift–Hohenberg equation*, J Math Chem **59**, 1474–1478 (2021).
2. Hyman J.M. and Nicolaenko B., *The Kuramoto–Sivashinsky equation: a bridge between PDE's and dynamical systems*, Physica D **18**, 113–126 (1986).

The Strategies and Tactics of an Intelligent Mathematical Modeling

V. P. Il'in

Institute of Computational Mathematics and Mathematical Geophysics SB RAS and Novosibirsk State University, Novosibirsk, Russia; ilin@sscc.ru

This research is devoted to the further modern development of the N. N. Yanenko's ideas on the deep learning and increasing efficiency of the various technological stages of large scale supercomputing experiments: geometrical and functional modeling, grid generation, space-time approximations, algebraic solutions, optimization approaches for inverse problems, post-processing and visualization of the numerical results, and decision making on the base of the analysis of obtained data. The considered integrated computational environment (ICE) presents an intelligent software tools for obtaining fundamental knowledge and investigation of the actual real processes and phenomena from the nature and industrial applications. The mathematical efficiency, high performance and robustness of the advanced approaches are provided by the different intelligent instruments for big data transforms as well as for automatical construction of algorithms and their mapping on the multiprocessor machine architectures with distributed and hierarchical shared memory. The ideal configuration of ICE is presented in the framework of the expert system or the base of active mathematical and/or information knowledge. The technical requirements for ICE include the extension of the set of models and algorithms, adaptation to the evolution of the computer platforms, wide reusing the external software, coordination of the various groups of developers. This features should support long life cycle of the product and its poste restante by the end users with wide class of professional backgrounds. Such intelligent computer technologies should play the role of the blood and/or lymph net systems for all sciences and industrial branches to provide the further progress and stable development of the civilization.

Numerical Study of Heat Modes of Laser Welding of Dissimilar Metals with an Intermediate Insert

V. I. Isaev[1], *A. N. Cherepanov*[1], *V.P. Shapeev*[1,2]

[1] *Khristianovich Institute of Theoretical and Applied Mechanics, 630090, Russia, Novosibirsk, Institutskaya st. 4/1*;

[2] *Novosibirsk State University, 630090, Russia, Novosibirsk, Pirogova st. 2*;
vshapeev@gmail.com

A 3D model of heat transfer in heterogeneous materials is developed. We used it for the numerical simulation of laser welding of metallic plates. We considered the welding of steel and titanium plates with an intermediate insert made of steel, copper, niobium, and titanium layers welded by explosion. They are used to prevent the formation of brittle intermetallic compounds in the weld that worsens the strength properties of the joint.

Welding is performed in two stages: first, the laser beam welds the joint of the steel plate with the steel outer layer of the insert, then the titanium plate is welded with the titanium outer layer of the insert.

The thickness of the layers in the insert is selected such that the refractory niobium layer does not melt. It prevents mixing of steel with titanium in the weld pool causing the formation of their intermetallic compounds. This improves the strength of the welded joint.

The model accounts for the key phenomena occurring during the complex physical process of laser welding: heat transfer in heterogeneous material, melting, evaporation, and solidification of materials. We used it to investigate the temperature fields in the plates and to predict the geometric parameters and the crystal structure of the weld. Finally, we defined the specifications for welding conditions (welding speed, laser power) and for the thicknesses of the layers in the insert allowing us to get a high quality weld joint.

We validated our simulations against corresponding experimental data and found them to be in a good agreement.

The research was carried out within the state assignment of Ministry of Science and Higher Education of the Russian Federation (project Nos. 121030500137-5 and AAAA-A19-119051590004 - 5).

Conservativeness of Invariant Finite-Difference Schemes

E. I. Kaptsov[1,2], *V. A. Dorodnitsyn*[2], *S. V. Meleshko*[1]

[1]*School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima, Thailand*; evgkaptsov@math.sut.ac.th, sergey@math.sut.ac.th

[2]*Keldysh Institute of Applied Mathematics, Moscow, Russia*; Dorodnitsyn@keldysh.ru, dorod2007@gmail.com

Methods for constructing finite-difference mathematical models based on symmetry are discussed. Group analysis methods make it possible to construct invariant difference schemes that preserve the basic geometric and qualitative physical properties of the original continuous models. Examples of invariant schemes for partial differential equations are given: wave equations, one-dimensional shallow water equations, and Green-Naghdi equations. The constructed schemes possess finite-difference analogs of the local conservation laws possessed by the original differential models.

The authors gratefully acknowledge financial support by Russian Science Foundation Grant 18-11-00238 "Hydrodynamics-type equations: symmetries, conservation laws, invariant difference schemes". E. I. K. acknowledges Suranaree University of Technology (SUT) and Thailand Science Research and Innovation (TSRI) for their support.

REFERENCES

1. Dorodnitsyn V. A., *Applications of Lie Groups to Difference Equations*, CRC Press, Boca Raton, 2011.
2. Dorodnitsyn V. A. and Kaptsov E. I., *Shallow water equations in Lagrangian coordinates: Symmetries, conservation laws and its preservation in difference models*, Communications in Nonlinear Science and Numerical Simulation, **89**, 105343 (2020).
3. Cheviakov A. F., Dorodnitsyn V. A. and Kaptsov E. I., *Invariant conservation law-preserving discretizations of linear and nonlinear wave equations*, Journal of Mathematical Physics, **61(8)**, 081504 (2020).
4. Dorodnitsyn V. A., Kaptsov E. I. and Meleshko S. V., *Symmetries, conservation laws, invariant solutions and difference schemes of the one-dimensional Green-Naghdi equations*, Journal of Nonlinear Mathematical Physics, **28**, 90–107 (2020).

General Solutions of Some Linear Equations with Variable Coefficients

O. V. Kaptsov, M. M. Mirzaokhmedov

Institute of Computational Modelling SB RAS, Akademgorodok, Krasnoyarsk, 660036, Russia; kaptsov@icm.krasn.ru

In this report we find general solutions to some classes of linear wave equations with variable coefficients. Such equations describe the oscillations of rods, acoustic waves, and also some models of gas dynamics are reduced to these equations. To construct general solutions, we employ special types of Euler-Darboux transformations, namely, Levi type transformations. These transformations are first order differential substitutions. For constructing each transformation, we need to solve two linear second order ordinary differential equations. The solutions of one of these equations are determined by the solutions of the other equations by means of a differential substitution and Liouville formula. In the general case, it is not easy to solve these ordinary differential equations. However, it is possible to provide some formula for the superposition of the transformation of Levy type.

Starting with a classical wave equation with constant coefficients and employing the found transformations, we can construct infinite series of equations possessing explicit general solutions. By means of Matveev method we obtain limiting forms of iterated transformations. We provide a series of particular examples of the equations possessing general solutions.

Using Analytical Continuation for Solving Nonlinear Free Boundary Problems.

E. A. Karabut[1], *E. N. Zhuravleva*[2]

[1] *Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences; Novosibirsk State University, Novosibirsk, Russia*; eakarabut@gmail.com

[2] *Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences; Novosibirsk State University, Novosibirsk, Russia*; zhuravleva_e@mail.ru

It is known that the Kirchhoff method allows one to find exact solutions for steady plane potential flows of an ideal fluid with a free boundary in the absence of the gravity force and surface tension. There are many jet flows of this type known today. However, if the problem is more complicated, for example, if the flow is ponderous, i.e., the gravity force is taken into account, or if the problem is unsteady, i.e., the time is taken into account, then there is no universal method for constructing exact solution in both cases.

In the present work, we propose the technique to obtain exact solutions basing on analytical continuation of the unknown function beyond the area of its definition. At first, by using conformal mapping the problem is formulated in the form of boundary-value problem in the fixed domain. As domains, we used a wedge with an apex angle α or the strip of unit width. After multiple turns around the wedge apex or after analytical continuation across the strip, various branches of the unknown function are related by an infinite system of differential equations.

It has been demonstrated that the system of equations becomes finite: if α/π is a rational number or if the function is periodic across the strip with a rational period.

By using the proposed technique, some of exact solutions have been obtained for stationary flows of heavy liquid, which turned out to be very close to high-amplitude gravity waves on the fluid surface. A new class of unsteady flows with a free boundary has been found.

The authors were supported by the Russian Foundation for Basic Research (project no. 19-01-00096).

REFERENCES

1. Karabut E. A. and Zhuravleva E. N., *Construction of exact solutions of the problem of the motion of a fluid with a free boundary using intinite systems of differential equations*, Theoretical and Mathematical Physics, **202**, 371–380 (2020).

Exact Solutions Having Diffusion Wave Type in Nonlinear Models of Thermal Conductivity, Filtration, and Diffusion

A. L. Kazakov[1], *M.-G. Lee*[2], *A. A. Lempert*[1]

[1] *Matrosov Institute for System Dynamics and Control Theory of Siberian Branch of Russian Academy of Sciences, Irkutsk, Russia*; kazakov@icc.ru, lempert@icc.ru

[2] *Chung Hua University, Hsin-Chu, Taiwan*; mglee1990@gmail.com

We consider the second-order nonlinear parabolic equation

$$u_t = [\Phi(u)]_{xx} + [\Psi(u)]_x + \Xi(u). \tag{1}$$

Here t, x are independent variables: t is time, x is a spatial variable; $u(t,x)$ is an unknown function, Φ, Ψ, Ξ are specified sufficiently smooth functions.

Equation (1) has a general form and isn't a mathematical model of any given process. However, its specific varieties describe many processes in continuum mechanics. The most commonly considered is the porous medium equation [1], which we get if $\Phi = Au^\alpha$, and $\Psi = \Xi \equiv 0$. It describes the filtration of an ideal gas in porous media and the radiant (nonlinear) thermal conductivity, so in Russian literature, it is called the nonlinear heat equation [2]. The second most known case is when $\Phi = Au^\alpha$, $\Psi \equiv 0$, and $\Xi = Bu^\beta$. Then Eq. (1) is called the generalized porous medium equation [1] or the nonlinear heat equation with a source [2]. The model allows one to take into account the inflow or sink of matter (energy). The case $\Phi = Au^\alpha$, $\Psi = Cu^\gamma$, $\Xi \equiv 0$ is less common, but it also arises in applications and describes the diffusion and convective transfer of energy and matter. It is called the convection-diffusion equation [3].

If function $\Phi'(u)$ is smooth and monotonic, the change of variable $v = \Phi'(u)$ brings Eq. (1) to the form

$$v_t = vv_{xx} + f(v)u_x^2 + g(v)v_x + h(v). \tag{2}$$

Let $f(0) \neq 0$, $h(0) = 0$. For Eq. (2) we consider the boundary condition

$$v|_{x=a(t)} = 0, \tag{3}$$

where $a(t)$ is a sufficiently smooth function, and $a'(0) \neq 0$. Problem (2), (3) it is typical for the scientific school of A. F. Sidorov [4]. Obviously, the trivial solution $v \equiv 0$ satisfies the problem. However, under certain natural additional conditions, problem (2), (3) also has a nontrivial solution, which changes sign when crossing the line $x = a(t)$ [5]. The positive part of this solution and the trivial solution form a diffusion (heat) wave, which propagates over zero background with a finite velocity.

The report will present exact solutions to Eq. (2), which have the diffusion wave type. Their construction is reduced to the integration of Cauchy problems for ordinary differential equations. Cauchy problems have a singularity multiplying the higher derivative, which is inherited from the original formulation. So, the classical existence and uniqueness theorems are not applicable in this case. We will discuss the existence and uniqueness of solutions, the procedure for their construction, and their properties. Some results were published in [3, 5], but the rests are new.

The reported study was funded by the Ministry of Science and Technology (MOST), grant 109-2923-E-216-001-MY3 and RFBR, research project 20-51-S52003.

REFERENCES

1. Vazquez J. L., *The Porous Medium Equation: Mathematical Theory*, Clarendon Press, Oxford (2007).
2. Samarskii A. A., Galaktionov V. A., Kurdyumov S. P. and Mikhailov A. P., *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyte, Berlin, 1995.
3. Kazakov A. L. and Spevak L. F., *On exact and approximate solutions to a problem with a singularity for the convection-diffusion equation* J. Appl. Mech. Tech. Phys., **62**(1), 18–26 (2021).
4. Sidorov, A. F., *Analytic representations of solutions of nonlinear parabolic equations of time-dependent filtration [porous medium] type*, Sov. Math., Dokl. **31**, 40–44 (1985).
5. Kazakov A. L., *Solutions to nonlinear evolutionary parabolic equations of the diffusion wave type*, Symmetry, **13**(5), 871 (2021).

Shape Gradient Method of Equilibrium-Constrained Optimization for Semi-linear Stokes–Brinkman–Forchheimer's Equation

V. A. Kovtunenko[1,2]

[1]*Institute for Mathematics and Scientific Computing, Karl-Franzens University of Graz, NAWI Graz, Heinrichstr.36, 8010 Graz, Austria*; victor.kovtunenko@uni-graz.at

[2]*Lavrent'ev Institute of Hydrodynamics, Siberian Division of Russian Academy of Sciences, 630090 Novosibirsk, Russia*;

A class of semi-linear optimization problems linked to variational inequalities is studied with respect to its shape differentiability. One typical example stemming from quasi-brittle fracture describes an elastic body with a Barenblatt cohesive crack under the inequality condition of non-penetration at the crack faces. Here we focus on the semi-linear model for a generalized Stokes–Brinkman–Forchheimer's equation under divergence-free and mixed boundary conditions describing the single-phase fluid flow in a porous medium. Based on the Lagrange multiplier approach, using suitable regularization and associated to adjoint operators due to Marchuk–Agoshkov–Shutyaev, an analytical formula for the shape derivative is derived from the Delfour–Zolesio theorem. The explicit expression contains both primal and adjoint states and is useful for finding descent direction of a gradient algorithm of numerical optimization to identify an optimal shape, e.g., from boundary measurement data as adopted in inverse problems.

The authors thank the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Programme (advanced grant No. 668998 OCLOC) and the Russian Foundation for Basic Research (RFBR) research project 18-29-10007 for partial support.

REFERENCES

1. Cakoni F. and Kovtunenko V. A., *Topological optimality condition for the identification of the center of an inhomogeneity*, Inverse Probl. **34**, 035009 (2018).
2. Ghilli D., Kunisch K. and Kovtunenko V. A., *Inverse problem of breaking line identification by shape optimization*, J. Inv. Ill-Posed Problems **28** (1), 119–135 (2020).
3. González Granada J. R., J. Gwinner J. and Kovtunenko V. A., *On the shape differentiability of objectives: a Lagrangian approach and the Brinkman problem*, Axioms **7**, 76 (2018).
4. González Granada J. R. and Kovtunenko V. A., *A shape derivative for optimal control of the nonlinear Brinkman–Forchheimer equation, J. Appl. Numer. Optim.* **3**, 243–261 (2021).
5. Itou H., Hirano S., Kimura M., Kovtunenko V. A. and Khludnev A. M. (Eds.), *Mathematical Analysis of Continuum Mechanics and Industrial Applications III: Proceedings of the International Conference CoMFoS18*, Mathematics for Industry **34**, Springer, Singapore (2020).
6. Kovtunenko V. A. and Kunisch K., *High precision identification of an object: Optimality-conditions-based concept of imaging*, SIAM J. Control Optim. **52**, 773-796 (2014).
7. Kovtunenko V. A. and Kunisch K., *Revisiting generalized FEM: a Petrov–Galerkin enrichment based FEM interpolation for Helmholtz problem*, Calcolo **55**, 38 (2018).
8. Kovtunenko V. A. and Ohtsuka K., *Shape differentiability of Lagrangians and application to Stokes problem*, SIAM J. Control Optim. **56**, 3668-3684 (2018).
9. Kovtunenko V. A. and Ohtsuka K., *Inverse problem of shape identification from boundary measurement for Stokes equations: shape differentiability of Lagrangian*, J. Inv. Ill-Posed Problems **29**, doi: 10.1515/jiip-2020-0081 (2021).

The Generalized Hermite Polynomials for the Burgers Hierarchy and Point Vortices

N. A. Kudryashov

Department of Applied Mathematics, National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 31 Kashirskoe Shosse, 115409 Moscow, Russia;
nakudr@gmail.com

Rational solutions of equations for the Burgers hierarchy are considered. Using self-similar variables this hierarchy is reduced to the family of nonlinear ordinary differenti al equations. Then the family is transformed to the hierarchy of non-autonomous linear differential equations by means of the Cole-Hopf formula. This hierarchy is a generalization of the second-order equation for Hermite polynomials. It is shown that every member of the hierarchy for ordinary differential equation has the solution in the form of polynomials. Properties of solutions of generalized Hermite equations in the form the special polynomials are studied. A recursion relation that can be used for finding corresponding poly nomials for every member is given. It is proved that the well-known property for Hermite polynomials connecting two polynomials can be used for all polynomials of the generali zed Hermite hierarchy. It is shown that the Cole-Hopf transformation is a direct consequence of the differential connection between two special polynomials in the hierarchy of Hermite equations. A derivation of the generalized Tkachenko equations is given for polynomials of the generalized Hermite hierarchy whose roots correspond to point vortices in the background

On Approximation Functionals to Minimal Splines

E. K. Kulikov[1], *A. A. Makarov*[2]

[1]*Department of Parallel algorithms, St. Petersburg State University, St. Petersburg, Russia*; egor.k.kulikov@gmail.com

[2]*Department of Parallel algorithms, St. Petersburg State University, St. Petersburg, Russia*; a.a.makarov@spbu.ru

We consider minimal coordinate splines [1]. These splines as a special case include well-known polynomial B-splines and share most of their properties (linear independence, smoothness, non-negativity, etc. Several types of approximation functionals to the system of minimal splines are constructed: a three-point quasi-interpolation method, an approximation of Sablonniere's and Grebennikov's types, and a system of de Boor-Fix Type functionals biorthogonal to minimal splines [2]. It is shown that in the case of polynomial generating vector functions for minimal splines obtained functionals coincide with well-known quasi-interpolants for B-splines [3, 4, 5]. We provide the results of numerical experiments of circular arc approximation that show how the error of approximation depends on the choice of approximation functionals and generating vector functions.

The reported study was funded by RFBR, project number 20-31-90095.

REFERENCES

1. Makarov A. A., *Construction of splines of maximal smoothness* Journal of Mathematical Sciences, **178**(6) 589–604 (2011).
2. Kulikov E. K. and Makarov A. A., *On de BoorFix Type Functionals for Minimal Splines* Topics in Classical and Modern Analysis (Applied and Numerical Harmonic Analysis), 211–225 (2019).
3. Kvasov B. I., *Methods of Shape-preserving Spline Approximation* World Scientific Publishing, Singapore, 356 p. (2000).
4. Sablonniere P., *Quadratic spline quasi-interpolants on bounded domains of* $\mathbb{R}^d$, *d = 1, 2, 3* Rendiconti del Seminario Matematico, **61**(3) , 229–246 (2003).
5. Grebennikov A. I., *Spline methods and solution of some ill-posed problems in approximation theory* (in Russian) Moscow University Press, 209 p. (1983).

Modelling Stationary Flows in Bounded Domain

D. E. Kukushkin[1], *N. I. Makarenko*[1,2], *V. P. Shapeev* [1,3]

[1]*Department of Mathematics and Mechanics, Novosibirsk State University, Novosibirsk, Russia*; d.kukushkin@g.nsu.ru

[2] *Lavrent'ev Institute of Hydrodynamics SB RAS, Novosibirsk, Russia*; makarenko@hydro.nsc.ru

[3] *Khristianovich Institute of Theoretical and Applied Mechanics SB RAS, Novosibirsk, Russia*; v.shapeev@g.nsu.ru

The problem under consideration concerns with internal waves in non-viscous environment generated by steady flows of incompressible fluid stratified by density. It is known that Euler equations for continuously stratified fluid can be reduced to a single quasi-linear elliptic Dubreil-Jacotin — Long equation for stream function. We use p-version of Collocations and Least Squares (CLS) method [1] to find numerical solution. This method produces satisfying qualitative and quantitative results by solving integral- and differential equations, in particular Navier – Stokes equations [1]. The CLS-method also has an advantage while we can find stream function explicitly as linear combination of basis functions, which means that we can find velocity components in every point of computational domain.

We consider in detail two model problems on 2D stationary flow in a bounded domain. The first one involves steady flow of inviscid homogeneous fluid confined between rigid bottom and top boundaries with semi-cylindric obstacle on the bottom. The second one deals with steady flow of non-homogeneous fluid between rigid bottom and top boundaries with Dirichlet boundary conditions on left and right sides of rectangular area.

REFERENCES

1. Shapeev V. P., Isaev V. I. and Idimeshev S. V., *The collocations and least squares method application to numerical solution of the Navier-Stokes equations*, Khristianovich Institute of Theoretical and Applied Mechanics, Russian Academy of Sciences, 2012.

Internal Waves in Two-Layer Stratified Flows

N. I. Makarenko, J. L. Maltseva, A. A. Cherevko

Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia; makarenko@hydro.nsc.ru

We consider an analytical model of internal waves propagating in a weakly stratified two-layer fluid. A new model equation involves the long-wave approximation suggested in [1,2] for the non-linear Dubreil-Jacotin - Long equation. The long-wave model describing travelling waves is constructed by means of scaling procedure with a small Boussinesq parameter. This model takes into account a slight density gradient in stratified layers which can be comparable with the density jump at the interface between layers. Parametric range of solitary wave is defined in the framework of considered mathematical model. It is demonstrated that non-linear wave regimes, including regimes of broad plateau-shaped solitary waves and internal fronts, can realize to be close to parametric domain of the Kelvin – Helmholtz instability. Such a marginal stability of finite-amplitude internal waves could explain the formation mechanism of a very long billow trains in abyssal flows.

This work was supported by the grant of the Russian Science Foundation (Project No 21-71-20039).

REFERENCES

1. Makarenko N. I., Maltseva J. L., Morozov E. G., Tarakanov R. Yu. and Ivanova K. A., *Internal waves in marginally stable abyssal stratified flow*, Nonlinear Processes Geophys., 2018, **25**, 659–669.
2. Makarenko N. I., Maltseva J. L., Morozov E. G., Tarakanov R. Yu. and Ivanova K. A., *Steady internal waves in deep stratified flow*, J. Appl. Mech. Techn. Phys., 2019, **60**(2), 248–256.

Method of Differential Constraints for Nonlinear Wave Problems

N. Manganaro

MIFT, University of Messina, Viale Ferdinando Stango D'Alcontres 31, 98166 Messina, Italy; nmanganaro@unime.it

Simple wave solutions are of great interest for nonlinear wave problems. Such a class of solutions are admitted by first order quasilinear hyperbolic systems in the homogeneous case and they are useful for solving different problems of interest in the applications as, for instance, Riemann problems. Unfortunately simple waves are not usually admitted by hyperbolic systems when dissipative effects are taken into account (non-homogeneous case). Within such a theroretical framework, in this talk, we would like to show how the well known Method of Differential Constraints can be used for studying different problems of great interest in nonlinear wave propagation [1]-[4].

REFERENCES

1. Yanenko N. N., *Compatibility theory and methods of integration of systems of nonlinear partial differential equation*, Proc. of the Fourth All-Union Math. Cong. (Leningrad) (Leningrad: Nauka), 247–252 (1964).
2. Manganaro N. and Meleshko S. V., *Reduction procedure and generalized simple waves for systems written in the Riemann Variables*, Nonlinear Dynamics, **30**(1), 87–102 (2002).
3. Manganaro N. and Pavlov M. V., *The Constant Astigmatism equation. New exact solution*, J. of Phys. A: Math. Theor., **47**(7), 075203, doi:10.1088/1751-8113/47/7/075203, 8pp (2014).
4. Meleshko S. V. and Shapeev V. P., *Nonisentropic solutions of simple wave type of the gas dynamics equations*, Journal of Nonlinear Mathematical Physics, **18**(1), 195–212 (2011).

Complete Set of Reciprocal Transformations of 2D Stationary Gasdynamics

S. V. Meleshko[1], *P. Siriwat*[2]

[1]*School of Mathematics, Institute of Science, Suranaree University of Technology, Nakhon Ratchasima, Thailand*; sergey@math.sut.ac.th
[2]*School of Science, Mae Fah Luang University, Chiang Rai, Thailand*; piyanuch.sir@mfu.ac.th

Equivalence transformations play one of the important roles in continuum mechanics. These transformations reduce the original equations to simpler forms. One of the classes of nonlocal equivalence transformations is the class of reciprocal transformations. Despite the long history of applications of such transformations in continuum mechanics, there is no method of obtaining them. Recently such a method was proposed. The method uses group analysis approach and it consists of similar steps as for finding equivalence group of transformations. The new method provides a systematic tool for finding classes of reciprocal transformations (reciprocal equivalence group of transformations). Similar to the classical group analysis this approach can be also applied for finding all discrete reciprocal transformations (not only composing a group). As an illustration the method is applied to the two-dimensional stationary gas dynamics equations. Equivalence group, reciprocal equivalence group and all discrete reciprocal transformations are found.

We are very thankful to Professor Colin Rogers for attracting our attention to reciprocal transformations.

Liouville Equation and Combinatorial Polynomials

D. Millionshchikov

Lomonosov Moscow State University and Gubkin Gas and Oil University, Moscow, Russia; mitia_m@hotmail.com

Consider the Liouville equation written in the form $u_{xy} = f(u)$ and two differential operators

$$D = \sum_{k=0}^{+\infty} u_{k+1}\frac{\partial}{\partial u_k},\ e^u X = \sum_{k=1}^{+\infty} D^{k-1}(e^u)\frac{\partial}{\partial u_k} = e^u \sum_{k=1}^{\infty} B_{k-1}\frac{\partial}{\partial u_k}$$

where $B_k = B_k(u_1, u_2, \dots, u_k)$ denote the k-th Bell polynomial defined recursively by

$$(D + u_1)B_k = B_{k+1}, \quad k \geq 0.$$

A polynomial $P(u_1, u_2, \dots)$ is called an integral of the Liouville equation if it anihilates the operator X. For instance $q_2 = u_2 - \frac{1}{2}u_1^2$ anihilates X.

Theorem[Shabat, Zhiber, 1979, [1]]. *The subalgebra $Ker\, X \subset \mathbb{K}[u_1, u_2, \dots]$ is isomorphic to the polynomial subalgebra $\mathbb{K}[q_2, q_3, \dots, q_k, \dots]$, where $q_k = D^{k-2}(q_2), k \geq 2$.*

One can remark that $Ker\, X$ is the eigen-subspace of the operator XD with $\lambda = 0$.

The defining equation of higher symmetries of the Liouville equation can be rewritten as

$$(D + u_1)XF = XDF = F.$$

It means, that a symmetry $F = F(u_1, u_2, \dots)$ is an eigen-vector of the operator XD with the eigen-value $\lambda = 1$.

Theorem[Zhiber, Shabat, 1979, [1]]. *An arbitrary symmetry F (an eigen-vector of XD with $\lambda = 1$) can be written in the form*

$$F = (D + u_1)Q,\ Q \in Ker\, X = \mathbb{K}[q_2, q_3, \dots].$$

Question: what about other eigen-spaces of the operator XD?

Theorem[2021, [2]] *1) The operator $(D+u_1)X = XD$ restricted to the subspace A_n of polynomials of weight $n \geq 2$ is diagonalizable;*

2) Its eigen-values are the following subset of non-negative triangular integers

$$\lambda_0 = 0, \lambda_1 = 1, \dots, \lambda_{n-2} = \frac{(n-2)(n-1)}{2}, \lambda_n = \frac{n(n+1)}{2}$$

(where $\lambda_{n-1} = \frac{(n-1)n}{2}$ is skipped);

3) An arbitrary eigen-vector P related to the eigen-value λ_k with $k \in \{0, 1, \dots, n-2, n\}$ can be written in the form

$$P = (D + u_1)(D + 2u_1)\dots(D + ku_1)F,$$

where F stands for some homogeneous polynomial of weight $(n - k)$ from $Ker\, X = \mathbb{K}[q_2, q_3, \dots]$.

REFERENCES

1. Zhiber A. V. and Shabat A. B., *The Klein-Gordon equation with nontrivial group*, (in Russian), Dokl. Akad. Nauk SSSR, **247**(5), 1103–1107 (1979).
2. Millionshchikov D V. and Smirnov S. V., *Characteristic algebras and integrable systems of exponential type*, Ufa Math. Journal, **13**:2, 44–73 (2021).
3. Millionshchikov D., *Lie Algebras of Slow Growth and Klein-Gordon PDE*, Algebras and Representation Theory, **21**:(5), 1037–1069 (2018).

Higher Order First Integrals of Autonomous Dynamical Systems

Antonios Mitsopoulos[1] *and Michael Tsamparlis*[2]

[1] *Faculty of Physics, Department of Astronomy-Astrophysics-Mechanics, University of Athens, Panepistemiopolis, Athens 157 83, Greece*; antmits@phys.uoa.gr

[2] *Faculty of Physics, Department of Astronomy-Astrophysics-Mechanics, University of Athens, Panepistemiopolis, Athens 157 83, Greece*; mtsampa@phys.uoa.gr

In general a system of differential equations is integrable if there exist 'enough' in number first integrals (FIs) so that its solution can be found by means of quadratures. Therefore the determination of the FIs is an important issue in order to establish the integrability of a dynamical system. In this work we consider holonomic autonomous dynamical systems defined by equations

$$\ddot{q}^a = F^a(q) \tag{1}$$

where the generalized force $F^a(q)$ has the form

$$F^a(q) = -A^a_{bc}(q)\dot{q}^b\dot{q}^c - Q^a(q).$$

We prove a Theorem which produces the FIs of any order of (1) in terms of the 'symmetries' of the geometry defined by the quantities A^a_{bc}, when they are considered to be connection coefficients of a non-metrical symmetric connection. We apply the Theorem to compute cubic FIs of various dynamical systems.

On Chains of Differential Equations

C. Muriel[1], *M. C. Nucci*[2], *J. L.Romero*[1]

[1]*Department of Mathematics, University of Cádiz, Cádiz, Spain*;
concepcion.muriel@uca.es
[2]*Dept. Math. Inf. Phys. Earth Sci., University of Messina, Italy*;

In this talk we collect and review some results on the integrability of sequences of autonomous ordinary differential equations. They are constructed by applying successively the differential operator $\mathbb{D}_g = D_x + g(u)$ to the function $u = u(x)$, where $g = g(u)$ is a given smooth function and D_x denotes the total derivative operator with respect to the independent variable x. The sequence (or chain) generated by g is denoted by $\mathbb{E}_g := \{\mathbb{D}_g^n u = 0\}_{n\in\mathbb{N}}$.

It was demonstrated in [1] that any of the equations in $\mathbb{E}_g$ admits the pair

$$(\mathbf{v}, \lambda) = \left(\partial_u, \frac{u_1}{u} - ug'(u)\right) \tag{1}$$

as $\mathcal{C}^\infty-$symmetry [2]. The main consequence is that the general solution of the nth-order equation in $\mathbb{E}_g$ is recursively constructed from the general solution of the first element $\mathbb{D}_g u = 0$, which is a first-order autonomous ODE. More specifically, if $\varphi_{[j]}(x)$ is the general solution of the jth element of the chain, then $\varphi_{[j+1]}(x)$ satisfies the first-order auxiliary equation $u_1 + g(u)u = u\varphi_{[j]}(x)$. There are many functions g for which this auxiliary equation can be easily integrated, as the Bernoulli-type equations that correspond to $g(u) = ku^m$. The well-known Riccati and Abel chains, corresponding to $m = 1$ and $m = 2$, respectively, are of this form [3, 4, 5].

In [6] the single $\mathcal{C}^\infty-$symmetry (1) was used to generate n generalized symmetries of the form $\mathbf{w}_{(n,i)} = \rho_{(n,i)}(u^{(n-1)})\partial_u$, for $1 \le i \le n$, of the nth element in the chain generated by $g(u) = ku^m$. As the main consequence, the functions $I_{(n,i)} = \rho_{(n,i)}/\rho_{(n,1)}$ for $2 \le i \le n$, are functionally first integrals of the nth-order equation that can be directly expressed in terms of the previous elements of the sequence.

For the Riccati chain we manage to get an additional generalized symmetry, which yields a complete set of first integrals, without any kind of integration. For the general case, the complete integration is achieved by using a Jacobi last multiplier [7] previously found in [1]. As a result, it is proved that the general solution of the nth element in the chain is of the form

$$u(t)^m = \frac{(T_{n-1}(t))^m}{km\displaystyle\int (T_{n-1}(t))^m\,dt + C_1},$$

where $T_{n-1}(t)$ is an arbitrary (normalized) polynomial of degree $n-1$. Several examples illustrates the main results.

C. Muriel gratefully acknowledges financial support by Junta de Andalucía research group FQM-377 and by FEDER/Ministerio de Ciencia, Innovación y Universidades-Agencia Estatal de Investigación/PGC2018-101514-B-I00 project.

REFERENCES

1. Muriel C. and Romero J. L., *λ-Symmetries of Some Chains of Ordinary Differential Equations*, Nonlinear Anal. Real World Appl., **16**, 191–201 (2014).
2. Muriel C. and Romero J. L., *New Methods of Reduction for Ordinary Differential Equations*, IMA J. Appl. Math., **66**, 111–125 (2001).
3. Andriopoulos K., Leach P. G. L. and Maharaj A., *On Differential Sequences*, AMIS, **5**, 525–546 (2011).
4. Cariñena J. F. , Lucas J. de and Rañada M. F., *A Geometric Approach to Integrability of Abel Differential Equations*, Int. J. Theor. Phys., **50**, 2114–2124 (2011).
5. Euler M., Euler N. and Leach P. G. L., *The Riccati and Ermakov-Pinney Hierarchies*, J. Nonlinear Math. Phys. , **14**, 290–310 (2007).
6. Muriel C. and Nucci M. C., *Generalized Symmetries, First Integrals, and Exact Solutions of Chains of Differential Equations*, Open Comm. Nonlinear Math. Phys., **1**, 1–16 (2021).
7. Nucci M. C., *Jacobi Last Multiplier and Lie symmetries: a Novel Application of an Old Relationship*, J. Nonlinear Math. Phys. , **12**, 284–304 (2005).

Symmetry Analysis of the Cylindrical Helmholtz Equation

M. Nadjafikhah[1], *A. Mahdipour–Sh.*[2]

[1]*Department of Pure Mathematics, School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, 16846-13114, Iran*; m_nadjafikhah@iust.ac.ir

[2]*Scientific Associate at University of Toronto Toronto, Ontario, Canada*; ali.mahdipour@gmail.com

In this paper, we present the point symmetry group of the three-dimensional homogeneous Helmholtz equation, when we consider the cylindrical coordinate system. In continuation, we present a complete set of functionally independent invariants of the equation along with the form of the general solution provided by these invariants. Finally, we find an optimal system of one-dimensional Lie subgroups of the full symmetry group.

Conservation Laws of the Relativistic Gas Dynamics Equations in Lagrangian Coordinates

W. Nakpim[1], *S. V. Meleshko*[2]

[1]*School of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, Thailand*; warisa@kku.ac.th

[2]*School of Mathematics, Institute of Science, Suranaree University of Technology, Nakhon Ratchasima, Thailand*; sergey@math.sut.ac.th

The research is focused on the analysis of the relativistic gas dynamics equations. The studied equations are considered in Lagrangian description, making it possible to find a Lagrangian such that the relativistic gas dynamics equations can be rewritten in a variational form. Such a Lagrangian is found in the paper. Complete group analysis of the Euler-Lagrange equation is performed. The found Lagrangian and the symmetries are used to derive conservation laws in Lagrangian variables by means of Noether's theorem. The analogs of the newly found conservation laws in Eulerian coordinates are presented as well.

Optimal System and Conservation Laws for Generalized Fisher Equation in Cylindrical Coordinates

S. Naseer[1], *A. Raza*[1], *F. D. Zaman*[1] *and A. H. Kara*[2]

[1]*Abdus Salaam School of Mathematical Sciences, Government College University, Lahore, Pakistan*; sonia.naseer19@sms.edu.pk, raza.ali@sms.edu.pk, f.zaman@sms.edu.pk
[2]*School of Mathematics, University of the Witwatersrand, Wits 2050, South Africa*; abdul.kara@wits.ac.za

Fisher Equation has been focus of many studies due to its applications in reaction diffusion process, biological and genetic modals and population dynamics. A generalized form of Fisher equation in cylindrical coordinates is studied here from Lie symmetry stand point. The optimal systems of symmetry generators are obtained and conservation laws using the multiplier approach are derived. In the end reductions using the similarity variables arising from Lie symmetries are presented.

Classical and Statistical Symmetries of Turbulence – the Basis of Turbulent Scaling Laws of Wall-Bounded Shear Flows for Arbitrary Moments

M. Oberlack[1,2]*, S. Hoyas*[3]*, S. Kraheberger*[1,2]*, F. Alcántara-Álvila*[3]*, J. Laux*[1]*, D. Klingenberg*[1,2]*, P. Hollmann*[1]*, M. Vallikivi*[4]*, M. Hultmark*[4]*, G. Bellani*[5]*, A. Talamelli*[5]*, S. Zimmerman*[6]*, J. Klewicki*[6]

[1]*Department of Mechanical Engineering, Technische Universität Darmstadt Otto-Bernd-Str. 2, Darmstadt 64287, Germany*; oberlack@fdy.tu-darmstadt.de

[2]*Centre for Computational Engineering, TU Darmstadt*;

[3]*Universitat Politècnica de València, Spain*;

[4]*Princeton University, USA*;

[5]*University of Bologna, Italy*;

[6]*The University of Melbourne, Australia*;

Using the symmetry-based turbulence theory, we derive turbulent scaling laws in wall-bounded shear flows for arbitrarily high moments of the flow velocity U_1. The key ingredients are the symmetries of classical mechanics, especially the scaling of space and time, and two statistical symmetries, which are not directly observed in Euler and Navier-Stokes equations. These symmetries are admitted by all complete theories of turbulence, i.e. the infinite hierarchy of moment and PDF equations and also by the famous Hopf functional equation. These symmetries provide a measure of intermittency and non-Gaussian behavior properties that have been investigated for decades for turbulence and are now subject to a rigorous description. Based on the above, in the near-wall region the symmetry theory predicts a log-law for the mean velocity ($n = 1$) and an algebraic law with the exponent $\omega(n-1)$ for moments $n > 1$. Hence, the exponent ω of the 2^{nd} moment determines the exponent of all higher moments. Most important, moments here always refer to the instantaneous velocities U and not to the fluctuations u'. For the core regions of both plane and round Poiseuille flows we find a deficit law for arbitrary moments n of algebraic type with a scaling exponent $n(\sigma_2 - \sigma_1) + 2\sigma_1 - \sigma_2$. Hence, the moments of order one and two with its scaling exponents σ_1 and σ_2 determine all higher exponents. All new group theoretical results are validated very well by a new plane Poiseuille flow DNS at $Re_\tau = 10^4$ and by pipe flow data from the CICLoPE (Uni Bologna) and Superpipe (Princeton) flow experiments at $Re_\tau = 2 * 10^3 - 3.8 * 10^4$. Thereby we find that σ_1 and σ_2 are almost identical, so that the exponents of all moments in this range are essentially constant, which corresponds to anomalous scaling.

Higher Order Symmetries of Underdetermined Systems of Partial Differential Equations and Noether's Second Theorem

P. J. Olver

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA;
olver@umn.edu

There are two well-known classes of partial differential equations that admit infinite hierarchies of higher order generalized symmetries:

1) linear and linearizable systems that admit a nontrivial point symmetry group;

2) integrable nonlinear equations such as Korteweg–de Vries, nonlinear Schrödinger, and Burgers'.

In this talk, I will introduce a new general class:

3) underdetermined systems of partial differential equations that admit an infinite-dimensional symmetry algebra depending on one or more arbitrary functions of the independent variables. An important subclass of the latter are the underdetermined Euler–Lagrange equations arising from a variational principle that admits an infinite-dimensional variational symmetry algebra depending on one or more arbitrary functions of the independent variables, which, according to Noether's Second Theorem, admit Noether dependencies. Examples include general relativity, electromagnetism, and parameter-independent variational principles.

Exact Solutions and Linearization of Ermakov-Pinney Equation Via the Nonlocal Transformation-Symmetry Approach

Ö. Orhan[1] *and Z. Pınar*[2]

[1]*Department of Engineering Science, Bandırma Onyedi Eylül University, Bandırma, Balıkesir, Turkey*; oorhan@bandirma.edu.tr

[2]*Department of Mathematics, Tekirdağ Namık Kemal University, 59030 Merkez, Tekirdağ, Turkey*; zpinar@nku.edu.tr

In this study, the nonlocal transformation, which is called Sundman transformation, and the Prelle-Singer method are used to solve ordinary differential equations. Firstly, the Ermakov-Pinney equation is classified by special functions to apply Sundman transformation. After this classification, the class of equations is determined. The suitable transformation for this class will be chosen, and the Sundman transformation pair is obtained using this appropriate transformation. Moreover, the first integrals of the Ermakov-Pinney equation are found by this transformation pair. Then, the Prelle-Singer approach is applied to Ermakov-Pinney Equation, and the first integrals, Lagrangian and Hamiltonian, of this equation, are derived. The exact solutions corresponding to the physically different cases are obtained by these first integrals and represented graphically.

Keywords: Nonlocal transformation pair, Differential equations, First Integral, Lagrangian, Hamiltonian, Ermakov-Pinney Equation.

REFERENCES

1. Ermakov V. P., Univ. Izv. Kiev, **20** (1880).
2. Pinney E., *The nonlinear differential equation* $\ddot{y}(x) + p(x)y + cy^{-3} = 0$, Proceedings of the American Mathematical Society, Oxford, **1** (1950).
3. Leach P. G., *Generalized Ermakov systems*, Physics Letters, **158** (1991).
4. Chandrasekar V. K., Senthilvelan M. and Lakshmanan M., *On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations*, Proc. R. Soc. A, (2005).

Lie Symmetries and Bohmian Inhomogeneous Cosmology

A. Paliathanasis

Institute of Systems Science, Durban University of Technology, Durban 4000, Republic of South Africa. ; anpaliat@phys.uoa.gr

We consider a family of inhomogeneous spacetimes known as silent universes with two scale factors. The gravitational field equations are constructed by an effective point-like Lagrangian. This Lagrangian function is used to write the field equations as a Hamiltonian system and derive the time-independent Schrodinger equation of quantum cosmology. We investigate the group properties of the Schrodinger equation and we calculate similarity solutions from Lie point symmetries. In the content of de BroglieBohm theory we determine the quantum correction in the semi-classical limit for the field equations. The quantum ponteial corresponds to new terms in the modified field equations and the new physical properties provided by the quantum potential are discussed.

About Convection of Compressible Gas

I. B. Palymskiy[1,2], *V. I. Palymskiy*[3]

[1]*Siberian State University of Telecommunications and Information Sciences, Novosibirsk 630102, Russia*; palymsky@yandex.ru

[2]*Siberian State University of Geosystems and Technologies, Novosibirsk 630108, Russia*;

[3]*Kutateladze Institute of Thermophysics, Siberian Branch of the Russian Academy of Sciences, Novosibrsk 630090, Russia*; palymsky1987@gmail.com

Rayleigh-Benard convection is a classical field of science, and in the overwhelming number of works convection is considered in an incompressible medium on the basis of the Boussinesq approximation [1]. However, the calculation of gas convection in regions with a height of several tens of centimeters and more requires taking into account its compressibility on the basis of the complete equations of gas dynamics. Similar tasks arise when considering the issues of explosion safety when transporting hydrocarbons through pipelines and storing them in tanks. However, such convection has been poorly studied due to the high rigidity of the system of equations due to the coexistence of fast thermoacoustic waves and slow convective motion [2].

In this work, by means of numerical simulation, we study the stability characteristics of the equilibrium Rayleigh-Benard convection regime in a compressible, viscous, and heat-conducting gas [2]. It is shown that, depending on the height of the convection region and the magnitude of the temperature gradient, various convection modes are realized - isobaric, adiabatic, and superadiabatic. Moreover, in the adiabatic regime, convection develops with stable stratification due to quasi-adiabatic processes. A diagram of convection regimes was constructed depending on the height of the region and the magnitude of the temperature gradient (Fig. 1). Analytical [3] and derived numerical data are shown with black solid line and dots. It is shown that for air under normal conditions the value of the critical height of the region Hcr, above which the isobaric convection regime is replaced by adiabatic with stable or superadiabatic with unstable in density stratification, is equal to 17.3 cm. An analytical formula for the critical height Hcr is obtained and for the selected gaseous medium, the critical height of the region increases with an increase in the absolute temperature according to the law of the fourth root.

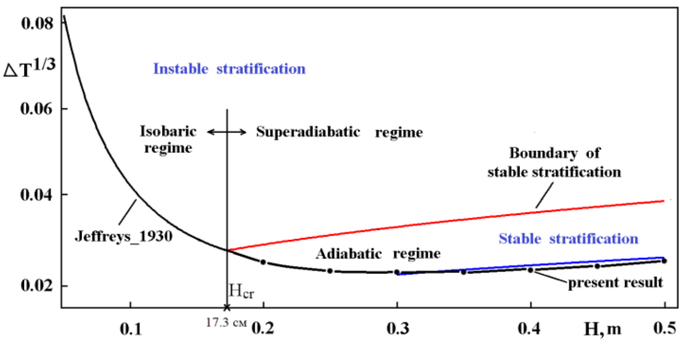


Figure 1: Dimensionless critical temperature difference

REFERENCES

1. Gershuni G. Z. and Zhukhovitskii E. M., *Convective Stability of Incompressible Fluids*, Israel Program for Scientific Translations, Jerusalem (1976).
2. Palymskiy I. B. and Trilis A. V., *Ob osobenostyach konvektsii v sgimaemom gase*, Trudy megdunarodnogo kongressa, Sib-Optika, 1–12 (2021) (in Russian).
3. Jeffreys H., *The stability of a compressible fluid heated from below*, Proc. Cambr. Phil. Soc., **26**, 170–172 (1930).

Egorov Hydrodynamic Type Systems

M. V. Pavlov

Laboratory of Mathematical Physics, P.N. Lebedev Physical Institute of Russian Academy of Sciences, Moscow, Russian Federation; mpavlov@itp.ac.ru

In this talk we consider the so called Egorov hydrodynamic type systems.

We discuss an integrability of these systems for any number of components.

So, the classical Hodograph Method can be appropriately extended from 2x2 to NxN quasilinear systems of this order.

The authors gratefully acknowledge financial support by RFBR grant 20-01-00157 A.

REFERENCES

1. Pavlov M. V. *Integrability of the Egorov hydrodynamic type systems*, Theor. Maths. Phys., **150**(2), 225–243 (2007).

Travelling Waves in 1D Strongly Inhomogeneous Media

E. Pelinovsky and T. Talipova

Institute of Applied Physics, Nizhny Novgorod, Russia; pelinovsly@appl.sci-nnov.ru

The problem of finding traveling waves in one-dimensional nonlinear and dispersive media is reduced to solving a system of ordinary differential equations; if the system order is high enough, the internal wave structure can be very complex, including random. If the medium is inhomogeneous, it is natural to expect the absence of solutions in the form of traveling waves due to the reflection and re-reflection effects. If, however, the medium parameters change slowly (in comparison with the wavelength), the reflection is weak, and by using asymptotic methods (WKB, geometric optics or acoustics) it becomes possible to construct an approximate solution in the form of a traveling wave with a variable amplitude and phase. For the media with a monotonic change in parameters, such solutions demonstrate the highest gain and the ability to transmit a signal over long distances without distortion.

It turns out to be possible to find exactly solutions in the form of traveling waves with variable amplitude and phase in highly inhomogeneous media under certain assumptions on the medium parameters. From the point of view of physics, such cases are possible for the so-called consistent media when there is no reflection even from a parameter jump. From the point of view of mathematics, such cases are possible when the governing equations in the corresponding variables can be reduced to equations with constant coefficients. It is the mathematical procedure for obtaining traveling waves in highly inhomogeneous media that is discussed in the report. We first demonstrate this procedure using the example of the classical linear wave equation with variable coefficient when it can be reduced to the Klein-Gordon equation with constant coefficients. This gives rise to an ordinary second-order differential equation for finding a variable coefficient (the sound speed), so that traveling waves exist in a wide class of inhomogeneous media. Then we demonstrate the effectiveness of this procedure in nonlinear hyperbolic problems as well as in the framework of Boussinesq systems.

Various examples of how this approach is applied to surface and internal waves in the ocean, sound waves in the Earth and Sun atmospheres, and magneto-hydrodynamics are considered.

The study was supported by grants RFBR 20-05-00162 (EP) and 19-05-00161 (TT).

Double Waves and Yanenko Equation

Z. Peradzynski[1], *G. Baghaturia*[2]

[1]*University of Warsaw, Warsaw, Poland*; zperadz@mimuw.edu.pl
[2]*Georgian Technical University, Tibilisi, Georgia*; nogel@gmail.com

One of many fields of scientific activity of academician Yanenko was searching for special solutions of PDEs, especially for solutions with degenerate hodograph of Euler equations of gas-dynamics [1-2]. The simplest example is given by so called double waves. For instance, transient 2-dimensional flow of an ideal gas is governed by a hyperbolic system of 3 equations for 3 unknown functions of (t, x, y). The solutions are mappings of R^3 into R^3 in general of rank 3. However, there are also solutions of rank 1 - simple waves and rank 2 - double waves. As noticed by Giese [3], double waves can be of hyperbolic or elliptic type. When searching for double waves one can determine at first the 2-dimensional hodograph surface S, which is the set of values of the double wave solution. The double wave hodographs are rather exceptional surfaces. It turns out that they have to satisfy some second order partial differential equation following from the degeneracy condition: rank of the solution = 2, which imposes extra constraints on the solution. Therefore appropriate compatibility condition must be satisfied. It turns out that systems of hydrodynamics type define two geometrical structures in the space of dependent variables; metric structure (or rather conformal structure) and a linear connection (independently of the metric). By appropriate definition, the metric structure can be properly selected so we have also the corresponding metric connection ∇^g. It turns out that the difference between these connections $G = \nabla - \nabla^g$ which is a tensor gives the measure of nonlinearity of the system (of potential flows). Moreover a 2-dimensional surface S is a double wave hodograph if and only if it satisfies the following second order (geometric) PDE [4-5]

$$H(x) = N(x) \quad \text{for every } x \in S$$

where $H(x)$ is the mean metric curvature of the surface, and $N(x) =< \tilde{G}, \tilde{g}^* >$ is expressed by the pull back of G to S and the contravariant tensor induced by g on the surface S. In [2-3] the above equation was presented for the case of gas-dynamics, although without giving it the geometrical meaning. In one of my conversations with N.N. Yanenko when discussing above equation, he said that he was aware that there must be some sort of geometry behind, especially the mean curvature of the surface. To acknowledge the contribution of Yanenko as well as M. Burnat [6] in the development of the theory of double and multiple waves solutions we can speak of Burnat-Yanenko equation.

REFERENCES

1. Pogodin I. I., Suchkov W. A. and Yanenko N. N., *O bieguscich volnach gazowoj dynamiki*, DAN, Sov. Union, **119**(3), (1958).
2. Sidorov A. F., Shapeev V. P. and Yanenko N. N., *The Method of Differential Constraints and Its Applications in Gas Dynamics*, Nauka, Novosibirsk (1984).
3. Giese J. H., *Compressible flows with degenerate hodographs*, Quart.Appl. Math, **2**, 237–242 (1951).
4. Peradzynski Z., *Geometry of Riemann waves* [In:] Debnath L. (ed.), *Advances in Nonlinear Waves*, Pitman, 244–285 (1985).
5. Peradzynski Z., *Geometry of nonlinear interaction in partial differential equations (in Polish). [In] Reports of the Institute of Fundamental technological research*, Warszawa, (1981).
6. Burnat M., *Hyperbolic double waves*, Bull. Acad. Polon. Sci. ser. Sci techn., **1**(876), (1968).

The Symmetries of the Fully Nonlinear Monge- Ampre Equation

Z. Pinar[1], *Ö. Orhan*[2]

[1]*Tekirda Namk Kemal University, Faculty of Arts and Science, Department of Mathematics, 59030 Merkez-Tekirda, Turkey*; zpinar@nku.edu.tr

[2]*Bandrma On Yedi Eyll University, Faculty of Engineering and Natural Science, Department of Engineering Science, Bandrma-Balkesir, Turkey*; oorhan@bandirma.edu.tr

The Lie group transformations have a major role in the applied science which is considered to obtain the fully nonlinear Monge-Ampre equation [1, 2]. For the reduced Monge-Ampre equation, Hermite approximation method and classical theory of differential equations are considered to obtain the analytical solutions. The obtained results have a major role in the literature so that the considered equation is seen in a large scale of applications.

REFERENCES

1. Udriste C. and Bila N., *Symmetry Lie Group of the Monge-Ampre Equation*, Balkan J. Geom. Appl. (BJGA)., **3**, 121–134 (1998).
2. Dean E. J. and Glowinski R., *Numerical methods for fully nonlinear elliptic equations of the Monge-Ampre type*, Comput. Methods Appl. Mech. Engrg., **195**, 1344–1386 (2006).

Reading N.N. Yanenko's Papers

V. P. Pukhnachev

Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia;
Novosibirsk State University, Novosibirsk, Russia; pukhnachev@gmail.com

Nikolai Nikolaevich Yanenko left a big creative heritage. The study of his publications stimulates the work in the directions he discovered. One of them is the study of double waves in the gas dynamics (Yu.Ya. Pogodin, V.A. Suchkov and N.N. Yanenko, 1958). The paper discusses the double waves equation in the case of a two-dimensional gas movement with a large indicator of polytropes γ and a decomposition of its solution with respect to inverse degrees of γ is constructed. Note that in the Tates well-known equation of the state, used to describe the barotropic water movements, $\gamma = 7.15$. N.N. Yanenko (1955) opened the class of weakly nonlinear systems of hyperbolic type equations. There are no strong discontinuities in their solutions although weak ones are allowed. It turned out that such a weakly nonlinear system occurs when describing the plane movement of the viscoelastic incompressible Maxwell medium near the critical point (N.P. Moshkin, V.V. Pukhnachev and Yu.D. Bozhkov, 2019). The paper discusses the axisymmetric version of this problem.

Symbolic Methods for Estimating the Sets of Solutions of Ordinary Differential Equations with Perturbations on a Finite Time Interval

A. N. Rogalev

Institute of Computational Modelling SB RAS, Krasnoyarsk, Russia;
rogalyov@icm.krasn.ru

In many works and reports of academician N.N. Yanenko, the problem statements are presented; they require computing the accuracy of numerical results that depend on the perturbations introduced. Such problems are equivalent to the problems of stability research. Problems of motion stability over a finite time interval are of considerable interest. This is also due to the fact that most of the methods for studying stability determine stability as $t \to \infty$. Practical stability over a finite time interval means that the solutions are uniformly bounded with respect to the set of initial values and the totality of disturbing influences. The report describes new results of using symbolic formulas of solutions sets for investigating practical stability. To estimate the boundaries of sets of solutions, we investigate and then use the property of injectivity (one-to-one) of solutions to ODEs. For linear systems of ODEs, the one-to-one property is a consequence of the Cauchy formula for the general solution. For nonlinear ODE systems that have unique solutions, the boundaries of the initial data domains pass into the boundaries of the solution domains at each specific moment. The class of such nonlinear ODE systems consists of systems with uniformly bounded solutions [1] (Lagrange stable).

The sets of solutions to ODE systems with initial data belonging to the regions of initial data have complex boundaries (boundary surfaces in a space of dimension less than or equal to n). For the boundaries (boundary surfaces) it is impossible to choose formulas of functions whose graphs coincide with the boundaries of the solution sets. As a result, it is possible to choose one of the algorithms – either describe the values of the boundary surfaces in a set of discrete points (on a grid), or compute the estimates of the maximum values of solutions in the directions of the coordinate axes, or compute the maximum values of solutions in any chosen direction. Preliminarily, it is useful to construct a regularization of estimates for the boundaries of the solution sets, passing to the linear approximation of the original system Regularization means obtaining information about a set of exact solutions. This regularization describes the values of compression/expansion in given directions, displacement along the time axis, and rotation through some angle of the set of solutions. In a sense, we can talk about estimating the deformation of the set of solutions in the linear approximation.

The use of symbolic formulas for solutions makes it possible to efficiently estimate the sets of solutions of ordinary differential equations with perturbations over a finite time interval. Examples of solving practical problems are given, confirming this efficiency.

This work was carried out within the framework of the state assignment of the Federal Research Center of the KSC SB RAS, project No. 0287-2021-0002

REFERENCES

1. Rogalev A. N., Rogalev A. A., Feodorova N.A. *Numerical computations of the safe boundaries of complex technical systems and practical stability* , J. Phys.: Conf. Ser., **1399**, 033112 (2019).

The Riemann Problem for Equations of Cold Plasma

O. S. Rozanova

Mechanics and Mathematics Faculty, Moscow State University, Moscow, Russia;
rozanova@mech.math.msu.su

The system of equations for the one-dimensional motion of cold plasma in the nonrelativistic case, as shown in [1], can be reduced to a hyperbolic system of two equations for the velocity $v(t,x)$ and the electric field $E(t,x)$, together with one equation for the density $n(t,x) > 0$:

$$v_t + vv_x = -E, \quad E_t + vE_x = v, \quad n = 1 - E_x, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}. \tag{1}$$

Since the derivatives of a smooth solution can go to infinity in a finite time [2], it makes sense to consider piecewise smooth initial data, the simplest example of which is the Riemann data

$$(v, E, n)|_{t=0} = (v^0_- + [v]^0\Theta(x),\ E^0_- + [E]^0\Theta(x), 1 + [E]^0\delta(x)), \quad [E]^0 \le 0, \tag{2}$$

where $\Theta(x)$ is the Heaviside function, constants $(v^0_\mp, E^0_\mp)$ are values to the left(right) of the jump, $([v]^0, [E]^0)$ are magnitudes of the jumps.

Since the initial data contain the delta function, the Riemann problem is singular and the Rankine-Hugoniot conditions cannot be written in the traditional form [3].

To construct the discontinuity, we write the system (1) in the divergent form

$$n_t + (vn)_x = 0, \qquad \left(\frac{nv^2}{2} + \frac{E^2}{2}\right)_t + \left(\frac{nv^3}{2}\right)_x = 0, \tag{3}$$

corresponding to the laws of conservation of mass and total energy (e,g., [4]).

The Riemann problem (3), (2) is completely non-standard and demonstrates new phenomena both in the rarefaction wave and in the singular shock wave.

The difficulty in constructing a solution is associated, in particular, with the fact that system (1) does not have a constant stationary state. Further, the system (1) for (v, E) is hyperbolic, but not strictly hyperbolic; it has a subclass of solutions distinguished by the condition $v^2 + E^2 = C^2$ with the given constant C. This leads to the non-uniqueness of the rarefaction wave for the Riemann problem; therefore, the question arises about the principles on which the "correct" solution can be distinguished.

When constructing a singular shock wave, a homogeneous conservative system of two equations (3) is used, but it includes three components of the solution, two of which are linked. Such a formulation has not been encountered before. The shock wave satisfies the so-called "supercompression" conditions, which are traditionally used to distinguish admissible singular shock waves [3].

REFERENCES

1. Chizhonkov E. V., *Mathematical Aspects of Modelling Oscillations and Wake Waves in Plasma*, CRC Press (2019).
2. Rozanova O. S. and Chizhonkov E. V., *On the conditions for the breaking of oscillations in a cold plasma*, Z. Angew. Math. Phys. **72**, 13 (2021).
3. Shelkovich V. M., *δ and δ'- shock wave types of singular solutions of systems of conservation laws and transport and concentration processes*, Russian Math. Surveys, **63** (3), 473–546 (2008).
4. Frolov A. A. and Chizhonkov E. V., *Application of energy conservation low in the cold plasma model*, Computational Mathematics and Mathematical Physics, **60** (3), 498–513 (2020).

Continuous Solutions to Sublinear Elliptic Problems

A. Seesanea

Sirindhorn International Institute of Technology, Thammasat University, Pathum Thani 12120, Thailand; adisak.see@siit.tu.ac.th

In this short talk, we shall focus on illustrating methods in potential theory and fixed point theory for the existence of positive continuous solutions to sublinear elliptic problems of the form

$$\begin{cases} -\Delta u = \sigma u^q + \mu & \text{in } \Omega, \\ u = f & \text{on } \partial^\infty \Omega. \end{cases}$$

Here σ and μ are positive Radon measures on an arbitrary domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), and f is a positive continuous function on the boundary $\partial^\infty \Omega$. If time permits, we may further discuss the uniqueness result and two-sided pointwise estimates of Brezis-Kamin type for such solutions. This talk is based on joint work with Kentaro Hirata (Hiroshima University).

Integrable Systems in Four Independent Variables from Contact Geometry

A. Sergyeyev

Mathematical Institute, Silesian University in Opava, Opava, Czech Republic;
artur.sergyeyev@math.slu.cz

According to Einstein's general relativity we live in a four-dimensional spacetime, and for this reason it is a longstanding challenge in mathematical physics to understand just how numerous and how diverse nonlinear partial differential systems in four independent variables that can be exactly solved in some sense can be and, in particular, whether an effective construction producing such systems is available, given that the number of previously known examples of the sort is rather limited.

In this talk we present such a construction and employ it to produce two new infinite families of sought-for examples, thus showing inter alia that there is significantly more of them than it appeared before.

Namely, we introduce a novel kind of Lax pairs related to contact geometry which yields a large new class of nonlinear partial differential systems in four independent variables (4D) integrable in the sense of soliton theory. The class in question contains inter alia two new infinite families of 4D integrable systems and a first known example of an integrable 4D system with a nonisospectral Lax pair which is algebraic rather than rational in the spectral parameter.

In particular, the following assertion holds:

Theorem. *Lax pairs* $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$, *where* $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$ *and* $\chi = \chi(x, y, z, t, p)$, *yield two infinite series of new 4D integrable systems for* $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$, *labelled by natural numbers* m *and* n, *for* f *and* g *given by the following formulas:*

1. $f = p^{n+1} + \sum\limits_{i=0}^{n} u_i p^i$, $g = p^{m+1} + \dfrac{m}{n} u_n p^m + \sum\limits_{j=0}^{m-1} v_j p^j$ *with* $\boldsymbol{u} = (u_0, \ldots, u_n, v_0, \ldots, v_{m-1})^{\mathrm{T}}$;
2. $f = \sum\limits_{i=1}^{m} \dfrac{a_i}{(p - u_i)}$, $g = \sum\limits_{j=1}^{n} \dfrac{b_j}{(p - v_j)}$ *with* $\boldsymbol{u} = (a_1, \ldots, a_m, u_1, \ldots, u_m, b_1, \ldots, b_n, v_1, \ldots, v_n)^{\mathrm{T}}$.

REFERENCES

1. Sergyeyev A., *New integrable (3+1)-dimensional systems and contact geometry*, Lett. Math. Phys. **108**, 359–376 (2018).
2. Sergyeyev A., *Integrable (3+1)-dimensional system with an algebraic Lax pair*, Appl. Math. Lett. **92**, 196–200 (2019).

Numerical Modeling of Cerebral Arterio-Venous Malformation Embolization Based on Clinical Data

T. S. Sharifullina, A. A. Cherevko, V. V. Ostapenko

Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences, 630090, Novosibirsk, Russia; igil@hydro.nsc.ru

Cerebral arteriovenous malformation (AVM) is a congenital brain vessels pathology, in which the arterial and venous blood channels are connected by tangles of abnormal blood vessels. The most preferred method of treating these pathologies is embolization - intravascular filling of the AVM vessel bundle with a special thickening composition (embolic agent) in order to block blood flow through them. This method of surgical intervention is widely used, but still in some cases it is accompanied by intraoperative rupture of the malformation vessels. In this work, this process is numerically modeled and an optimization algorithm for embolization is built.

To describe the embolization process, a combined model is proposed, in which, along with the flow of blood and embolic agent in the AVM, the redistribution of blood to the surrounding healthy vessels is taken into account. The embolization process is modeled as a two-phase filtration process of immiscible incompressible fluids, where the displaced phase is blood, and the displacing phase is the embolic agent; for this, an equation of the Buckley-Leverett type is used, which is solved numerically using a monotonic modification of the CABARET scheme [1]. The blood flow entering the AVM changes during the operation due to the redistribution of blood to adjacent healthy vessels; this effect is taken into account in the model by introducing additional algebraic and integral relations.

When studying the problem of optimal embolization, the geometric and filtration AVM characteristics were used, built on the basis of clinical data obtained during the monitoring of hemodynamic parameters during neurosurgical operations at the National Medical Research Center named after academic E.N. Meshalkin [2].

The main goal of this work is to find the optimal scenario for arteriovenous malformation embolization from the safety and effectiveness of the procedure point of view. The objective functional and the constraints arising in such an optimal control problem are selected in accordance with medical indications. The control is a time-dependent function that determines the volumetric flow rate of the embolic agent at the AVM input. The problem of embolization optimal control is formulated and solved for a special law of embolic agent supply.

The work was supported by the Russian Science Foundation (grant number 20-71-10034).

REFERENCES

1. Ostapenko V. V. and Cherevko A. A., *Application of the CABARET scheme for calculation of discontinuous solutions of the scalar conservation law with nonconvex flux*, Dokl. Phys., Pleiades Publishing Ltd., **62**(10), 470-474 (2017).
2. Khe A. K. et al., *Monitoring of hemodynamics of brain vessels*, J. App. Mech. Tech. Phys., **58**(5), 763-770 (2017).

Submodels and Exact Solutions of the Gas Dynamics Equations with State Equation of a Special Form

D. T. Siraeva

Laboratory of Differential Equations of Mechanics, Mavlyutov Institute of Mechanics UFRC RAS, Ufa, Russia; sirdilara@gmail.com

The gas dynamics equations with the state equation of a general form have the following symmetries: space translations, time translation, rotations, Galilean translations, uniform dilatation. In this investigation the state equation is a pressure equal to the sum of two functions - the first function depends on density, and the second function depends on entropy [1]. Such system of equations has additional symmetry — pressure translation. The system admits a 12-dimensional Lie algebra. An optimal system of dissimilar subalgebras of the 12-dimensional Lie algebra was constructed in [2]. Invariant submodels of rank 3, 2, and 1 are calculated for 1-, 2-, and 3-dimensional subalgebras. Exact solutions were found for some submodels [3, 4]. The motion of particles and volumes according to some exact solutions is considered.

The author was supported by the Russian Foundation for Basic Research (project no. 18-29-10071) and partially from the Federal Budget by the State Target (project no. 0246-2019-0052).

REFERENCES

1. Ovsyannikov L. V., *The "podmodeli" program. Gas dynamics*, Journal of Applied Mathematics and Mechanics, 1994. Vol. 58, No. 4. Pp. 601–627. Doi:10.1016/0021-8928(94)90137-6
2. Siraeva D. T., *Optimal system of non-similar subalgebras of sum of two ideals*, Ufa Mathematical Journal, 2014. Vol. 6, No. 1. Pp. 90–103. Doi:10.13108/2014-6-1-90
3. Siraeva D. T. and Khabirov S. V., *Invariant submodel of rank 2 on subalgebra of translations linear combinations for a hydrodynamic type model*, Chelyabinsk Physical and Mathematical Journal, 2018. Vol. 3, No. 1. Pp. 38–57.
4. Siraeva D. T., *Two invariant submodels of rank 1 of the hydrodynamic type equations and exact solutions*, Journal of Physics: Conference Series, 1666 012049 (2020). Doi:10.1088/1742-6596/1666/1/012049

On Difference Schemes for the Second-Order Differential-Algebraic Equations

L. S. Solovarova[1], *T. D. Phuong*[2]

[1]*Matrosov Institute for System Dynamics and Control Theory of SB RAS, Irkutsk, Russia*; soleilu@mail.ru

[2]*Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam*; tdphuong@math.ac.vn

We consider the problem

$$A(t)x''(t) + B(t)x'(t) + C(t)x(t) = f(t), t \in [0, 1], \tag{1}$$

$$x(0) = x_0, x'(t)|_{t=0} = x_0', \tag{2}$$

where $A(t), B(t), C(t)$ are $(n \times n)$-matrices, $f(t)$ and $x(t)$ are the given and unknown n-dimensional vector functions $x_0, x_0' \in R^n$. Here

$$det A(t) \equiv 0. \tag{3}$$

System (1) with condition (3) is called as second-order differential-algebraic equations (DAEs2). It is assumed that the initial conditions (2) are consistent, that is, the problem under consideration has a solution. By the solution we mean any differentiable vector function that turns (1) into an identity and satisfies conditions (2).

The main approach to solving the second-order DAEs is that the original problem is rewritten as the first-order DAE, using the new vector-function $y(t) = (x^{\top'}(t), x^{\top}(t))^{\top}$ [1]. This transformation has drawbacks, since it doubles the dimension of the obtained problem.

We propose methods based on the idea from [2] for direct numerical solution of problem under consideration.

The reported study was funded by RFBR and VAST according to the research project 20-51-54003.

REFERENCES

1. Mehrmann V. and Shi C., *Transformation of high order linear differential-algebraic systems to first order*, Numer. Algorithms, **42**, 281–307 (2006).
2. Bulatov M. V., Lee M.-G. and Solovarova L. S., *On first- and second-order difference schemes for differential-algebraic equations of index at most two*, Comput. Math. Math. Phys., **50**(11), 1808–1817 (2010).

Scalar Description of Three-Dimensional Flows of Incompressible Fluid

Y. A. Stepanyants[1,2]

[1]*School of Sciences, University of Southern Queensland, West St., Toowoomba, QLD, 4350, Australia*; Yury.Stepanyants@usq.edu.au

[2]*Department of Applied Mathematics, Nizhny Novgorod State Technical University n.a. R.E. Alekseev, 24 Minin St., Nizhny Novgorod, Russia*;

Essential progress in the investigation of flows of an incompressible fluid is usually achieved with the help of scalar functions, e.g., the velocity potential or stream-function. Indeed, flow description by means of one scalar function is much simpler than the description based on the three-dimensional vector field. Many interesting and physically important problems were solved in this way. However, the traditional usage of the velocity potential or stream-function is restricted by certain assumptions – in the former case, the flow is assumed to be ideal and potential, whereas in the latter case the flow may be viscous, but consisting of two components only with only one component of the vorticity. Such restrictions essentially bound a range of applicability of the traditional approaches.

Here we propose another approach, also based on the introduction of only one scalar function dubbed the quasi-potential. However, it is shown that with the quasi-potential a wide new class of non-stationary three-dimensional flows can be described. This class of flows includes both the potential and vortex flows as the particular cases. In the latter case, the corresponding vorticity field may consist of two components, in general. Characteristic features of such flows are described in detail. Particular examples of flows are presented in the explicit form. We also derive the Bernoulli integral for this class of flows and compare it against the known Bernoulli integrals for the potential flows or 2D stationary vortical flows of an inviscid fluid. We show that the Bernoulli integral for this class of fluid motion possesses unusual features: it is valid for the vortical non-stationary motions of a viscous incompressible fluid. A further non-trivial generalisation can be done for the flows in curvilinear coordinate frames, for example, in the cylindrical or spherical coordinate frame.

The author gratefully acknowledges financial support by the State task program in the sphere of scientific activity of the Ministry of Science and Higher Education of the Russian Federation (project No. FSWE-2020-0007) and the grant of President of the Russian Federation for the state support of Leading Scientific Schools of the Russian Federation (grant No. NSH-2485.2020.5).

REFERENCES

1. Stepanyants Y. A. and Yakubovich E I., *Scalar description of three-dimensional vortex flows of incompressible fluid*, Doklady Physics, **56**(2), 130–133 (2011).
2. Stepanyants Y. A. and Yakubovich E I., *The Bernoulli integral for a certain class of non-stationary viscous vortical flows of incom-pressible fluid*, Stud. Appl. Math., **135**, 295–310 (2015).
3. Ermakov A. M. and Stepanyants Y. A., *Description of nonlinear vortical flows of incompressible fluid in terms of quasi-potential*, Physics, **3** (2021) – in press.

On Differential-Invariant Solutions of the Navier-Stokes Equations with Respect to one Four-Dimensional Group

A. A. Talyshev

Department of Mechanics and Mathematics, Novosibirsk State University, Novosibirsk, Russia; tal@academ.org

In this paper, we consider the differentially-invariant solutions of the equations Navier-Stokes with respect to the four-dimensional group generated by operators

$$\partial_x,\ \partial_y,\ t\partial_x + \partial_u,\ t\partial_y + \partial_v. \tag{1}$$

Partially invariant solutions of defect 2 and rank 2 with respect to this group for the Navier-Stokes equations are described in [2] and for the equations gas dynamics in the work [3].

Differential-invariant solutions are a generalization of invariant and partially invariant solutions [1]. Each differentially-invariant solution is characterized by the sequence dimensions of orbits $d_0, d_1, \ldots$ of this solution in extended spaces [4], [5], where d_j is the dimension of the orbit of the solution in the j-extended space.

For the algebra (1), the following sequence variants are possible: (6,6), (6,7,7), (6,7,8,8), (7,7), (7,8,8), (8,8). The first three correspond to partially invariant solutions of defect 2, following two partially invariant solutions of defect 3. In this paper, we consider in detail variants (6,6) and (7,7).

REFERENCES

1. Ovsianikov L. V., *Group Analysis of Differential Equations*, Academic Press, New York (1982).
2. Meleshko S. V. and Pukhnachev V. V, *On a class of partially invariant solutions of the Navier-Stokes equations*, Applied Mechanics and Technical Physics, **40**(2), 24–33 (1999).
3. Ovsianikov L. V., Chupakhin A. P. *Regular Partially Invariant Submodels of Gas Dynamics Equations*, Applied Mathimatics and Mechanics, **60**(6), 990–999 (1996).
4. Talyshev A. A., *Automorphic Systems and Differential-Invariant Solutions*, Regularity and Stochasticity of Nonlinear Dynamical Systems, Nonlinear Systems and Complexity, Springer International Publishing AG, 283–308 (2018).
5. Talyshev A. A., *Differential-Invariant Solutions*, Journal of Mathematical Sciences **230**, 167-174 (2018).

All Exact Symmetries of Higher-Order ODEs are Stable

M. R. Tarayrah [1], *Alexei F. Cheviakov* [2]

[1] *Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada*; mrt566@usask.ca

[2] *Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada*; shevyakov@math.usask.ca

The framework of Baikov-Gazizov-Ibragimov approximate symmetries has proven useful for many examples where a small perturbation of an ordinary differential equation (ODE) destroys its local symmetry group [1, 2]. For second and higher-order ODEs: some original symmetries of the unperturbed model

$$y^{(n)} = f_0(x, y, y', ..., y^{(n-1)}), \quad n \geq 2 \tag{1}$$

can be unstable, that is, they are not inherited as nontrivial approximate point symmetries of a perturbed ODE

$$y^{(n)} = f_0(x, y, y', ..., y^{(n-1)}) + \epsilon f_1(x, y, y', ..., y^{(n-1)}) + o(\epsilon). \tag{2}$$

We show that the unstable point symmetries of the unperturbed ODE (1) correspond to higher-order approximate symmetries of the perturbed ODE (2), and can be systematically computed:

Theorem 1. *For each exact point or local symmetry of an unperturbed ODE* (1), *there is an approximate symmetry of the perturbed ODE* (2), *with the approximate symmetry component being of order at most* $n-1$.

As an application, we consider a fourth-order Boussinesq reduction ODE

$$y^{(4)} + y'' - \epsilon\left(2yy'' + 2y'^2\right) = 0. \tag{3}$$

REFERENCES

1. Ibragimov, N. H., *CRC handbook of Lie group analysis of differential equations*, CRC Press, (1995).
2. Baikov, V. A., Gazizov R. K. and Ibragimov N. H., *Approximate symmetry and formal linearization*, Journal of Applied Mechanics and Technical Physics 30.2: 204–212 (1989).

Lyapunov Instability of the Polymeric Fluid Flow in Channel (Channel Walls are Perforated)

D. L. Tkachev

Sobolev Institute of Mathematics, Novosibirsk, Russia; tkachev@math.nsc.ru

We study linear stability by Lyapunov of the previously obtained stationary flow of viscoelastic incompressible polymeric fluid in an infinite plane channel with perforated walls. To describe the movement of the polymeric medium we choose rheological Pokrovski-Vinogradov model, which is good enough in catching qualitative properties of flows in observed experiments [1].

It turns out that there are solutions in an exponential form with growing exponent in perturbation classes of the base solution periodic with respect to the spatial variable.

This work continues the study of linear stability of stationary of flows for different solutions of the Pokrovski-Vinogradov model and its generalizations to the nonisothermic case and to the case when medium is influenced by the uniform external magnetic field and model taking into account nonisothermic electroconvection of weakly isolating polymeric fluid [2, 3, 4, 5, 6, 7, 8, 9].

The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. 0314-2019-0013) and additionally supported by the RFBR, project number 19-01-00261a.

REFERENCES

1. Pyshnograi G. V., Pokrovski V. N., Yanovski Yu. G., Obraztsov I. F. and Karnet Yu. N., *Defining equation for nonlinear viscoelastic (polymeric) mediums in zero approximation by parameters of molecular theory and results for shear and stretch*, Dokl. Akad. Nauk SSSR, **339**(5), 612–615 (1994).
2. Blokhin A. M., Yegitov A. V. and Tkachev D. L., *Linear instability of solutions in a mathematical model describing polymer flows in an infinite channel*, Computational Mathematics and Mathematical Physics, **55** (5), 848–873 (2015).
3. Blokhin A. M., Yegitov A. V. and Tkachev D. L., *Asymptotics of the Spectrum of a linearized problem of the stability of a stationary flow of an incompressible polymer fluid with a space charge*, Computational Mathematics and Mathematical Physics, **56**(1), 102–117 (2018).
4. Alexander Blokhin, Dmitry Tkachev and Aleksey Yegitov, *Spectral asymptotics of a linearized problem for an incompressible weakly conducting polymeric fluid*, ZAMM (Z. Angrew. Math. Mech.), **98**(4), 589–601 (2018).
5. Blokhin A. M. and Tkachev D. L., *Stability of the Poiseuille-type flow for a MHD model of an incompressible polymeric fluid*, Europian Journal of Mechanics / B. Fluids, **80**, 112–121 (2020).
6. Blokhin A. M. and Tkachev D. L., *Stability of Poiseuille-type flows in an MHD model of an incompressible polymeric fluid*, Sbornik:Mathematics, **211**(7), 901–921 (2020).
7. Blokhin A. M. and Tkachev D. L., *Stability of the Poiseuille-type flows for an MHD model of an incompressible polymeric fluid*, Journal of Hyperbolic Differential Equations, **16**(4) 1–25 (2019).
8. Blokhin A. M. and Tkachev D. L., *Linear asymptotic instability of a stationary flow of a polymeric medium in a plane channel in the case of periodic perturbations*, Journal of Applied and Industrial Mathematics, **8**(4), 467–478 (2014).
9. Blokhin A. M. and Tkachev D. L., *MHD model of incompressible polymeric fluid. Linear instability of the resting state*, Complex Variables and Elliptic Equations, **66**(6–7), 929–944 (2021).

Uniform Approximation of an Impulsive Differential System by Using Piecewise Constant Arguments on the Halfline

R. Torres

Instituto de Cs. Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile; ricardo.torres@uach.cl

A. Myshkis was the first to mention differential equations of the form $x'(t) = f(t, x(t), x(r(t)))$; where $r(t)$ is a discontinuous argument; for example, $r(t) = [t]$, the integer part function.
M.U. Akhmet in order to generalize the situation presented before, introduced the equations of the form $x'(t) = f(t, x(t), x(\gamma(t)))$, where $\gamma(t)$ is a *piecewise constant argument of generalized type*. This is, given $(t_k)_{k\in\mathbb{Z}}$ and $(\zeta_k)_{k\in\mathbb{Z}}$ such that $t_k < t_{k+1}, \forall k \in \mathbb{Z}$ with $\lim_{k\to\pm\infty} t_k = \pm\infty$, $t_k \le \zeta_k \le t_{k+1}$ and $\gamma(t) = \zeta_k$ if $t \in I_k = [t_k, t_{k+1})$. These equations were called *Differential Equations with Piecewise Constant Argument of Generalized Type* (*DEPCAG*). It is very remarkable that, despite the discontinuous deviating argument, they have continuous solutions. At the end of the intervals $I_k = [t_k, t_{k+1})$, they define a difference equation. This important fact gives to these equations the name of *hybrids*. Hence, it is necessary to take into account discrete and continuous dynamics.
If in the DEPCAG case, continuity at the endpoints of the intervals $I_k = [t_k, t_{k+1})$ is not required, a new type of equations is defined. They are called *Impulsive Differential Equations with Piecewise Constant Argument of Generalized Type* (*IDEPCAG*)

$$\begin{aligned} x'(t) &= f(t, x(t), x(\gamma(t))), \quad t \neq t_k \\ \Delta x(t_k) &= Q_k(x(t_k^-)), \qquad t = t_k, \end{aligned} \tag{1}$$

where $\Delta x(t_k) = x(t_k) - x(t_k^-)$, $x(t_k^-) = \lim_{\substack{t\to t_k \\ t<t_k}} x(t)$ exists $\forall t_k$ with $k \in \mathbb{N}$, and $x(t_k)$ is defined by $x(t_k) = x(t_k^-) + Q_k\left(x(t_k^-)\right)$.
Consider the differential equation with deviated argument

$$x'(t) = f\left(\left[t/h\right]h, x\left(\left[t/h\right]h\right)\right), \tag{2}$$

where $x_0 = x(kh)$, with $k \in \mathbb{Z}$, $h \in]0, \infty[$ fixed and $[\cdot]$ denotes the integer part function. We note that $[t/h]\,h = kh$ if $t \in I_k = [kh, (k+1)\,h)$. This function is a case of a piecewise constant function and it has jump discontinuities at the points $\mathbb{Z}h = \{kh : k \in \mathbb{Z}\}$. Then, if we set $t_k = kh$, integrating (2) in $I_k = [t_k, t_{k+1})$ and assuming continuity at $t = t_{k+1}$ we have $x(t_{k+1}) = x(t_k) + hf(t_k, x(t_k))$. So, (2) can be seen as an approximating for $x'(t) = f(t, x(t))$, recovering the Euler's scheme. Using the step function $\gamma(t) = [t/h]\,h$ and some stability hypothesis, we will approximate the solution of the following impulsive system

$$\begin{aligned} y_i'(t) &= -a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(y_j(t)) + c_i(t), \quad t \neq t_k \\ \Delta y_i(t_k) &= -q_{i,k}y_i(t_k^-) + e_{i,k} + I_{i,k}(y(t_k^-)), \qquad t = t_k, \end{aligned}$$

by the impulsive system with piecewise constant argument

$$\begin{aligned} z_i'(t) &= -a_i(t)z_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(z_j(\gamma(t))) + c_i(t), \qquad t \neq \gamma(t_k) \\ \Delta z_i(\gamma(t_k)) &= -q_{i,k}z_i(\gamma(t_k)^-) + e_{i,k} + I_{i,k}(z(\gamma(t_k)^-)) \quad t = \gamma(t_k). \end{aligned}$$

The approximation will be uniform in $[\tau, \infty)$. I.e $\sup_{t\in[\tau,\infty)} |y(t) - z(t)| \to 0$ as $h \to 0$, with error of approximation $\sup_{t\in[\tau,\infty)} |y_i(t) - z_i(t)| \le \exp\{-\sigma_2(t-\tau)\}\,K(h, \sigma, \tau)$, $\sigma_2 > 0$.

This is a joint work with M. Pinto (U. de Chile) and A. Paliathanasis (Durban U. of Technology).

REFERENCES

1. Torres R., Pinto M., Castillo S. and Kostić M., *An uniform approximation of an impulsive CNN-Hopfield type system by an impulsive differential equation with piecewise constant argument of generalized type on* $[\tau, \infty)$, Acta Appl. Math. **171**-8 (2021). https://doi.org/10.1007/s10440-020-00373-3

Discrete Orthogonal Polynomials: Detection of Anomalies of Time Series and Boundary Effects of Polynomial Filters

S. P. Tsarev

Siberian Federal University, Krasnoyarsk, Russia; sptsarev@mail.ru

We describe a new result in the classical theory of univariate discrete orthogonal polynomials: extremely fast decay of their values near the interval boundary for polynomials of sufficiently high degree. This effect dramatically differs from the behavior of much more popular in mathematical curricula continuous orthogonal polynomials.

The practical importance of this new result for the theory of discrete polynomial filters (widely applied for detection of anomalies of time series of measurements) is demonstrated on the practical example of detection of outliers and small discontinuities in the publicly available GPS and GLONASS trajectories.

Discrete polynomial filters, on one hand, can detect very small anomalies in sparse time series (with amplitude of order 10^{-11} relative to the typical values of the time series). On the other hand our general result limits sensitivity of polynomial filters near the boundary of the time series.

The main problem in practical applications of the discussed method is numerical instability of construction of the discrete orthogonal polynomials of high degree. We present a simple and robust way of numerical computation of discrete orthogonal polynomials (Hahn polynomials).

The authors gratefully acknowledge financial support by Krasnoyarsk Mathematical Center and the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement No. 075-02-2021-1388).

REFERENCES

1. Tsarev, S. P. and Kytmanov A. A., *Discrete orthogonal polynomials as a tool for detection of small anomalies of time series: a case study of GPS final orbits*, arXiv preprint arXiv:2004.00414 (2020). https://arxiv.org/abs/2004.00414

On Some Methods of Reducing Nonlinear Partial Differential Equations to Systems of Ordinary Differential Equations

O. N. Ulyanov, L. I. Rubina

Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, Ekaterinburg, Russia; uon@imm.uran.ru

The study of a mathematical model specified in the form of a nonlinear partial differential equation or a system of nonlinear partial differential equations can be carried out in different ways and methods. Among them, of course, we should mention the method of differential constraints [1], methods of group analysis of differential equations [2], the Clarkson-Kruskal method [3]. As is known, there are no general methods for analyzing such models. When solving practical problems, one has to develop various original techniques. The report discusses three ways to reduce nonlinear partial differential equations to systems of ordinary differential equations (systems of ODEs). These methods are based on a system of equations of characteristics for a certain first-order partial differential equation. Let's call this equation as basic equation.

In the first method, the "basic equation is set. On the left-hand side, it contains only the independent variables of the original equation and the first derivatives of the solution to the original equation. On the right-hand side of the basic equation, an arbitrary function is specified that depends only on the solution of the original equation. An arbitrary function allows one to construct such a system of ODEs — a system of equations for the characteristics of a given basic equation, for which the original equation is the first integral.

In the second method, the part of the original equation containing only first-order derivatives is selected as the basic equation, if such a part is present in the original equation.

In the third method, the first-order equation is selected as the basic equation, which must be satisfied by the function that defines the level surface of the solution to the original equation.

When one or another method is selected and the corresponding basic equation is selected, a system of characteristic equations is written out for the "basic" first-order partial differential equation. Note that it describes the change in independent variables, the solution to this equation, and the first derivatives of the solution to the basic equation along the characteristics. The system is supplemented by equations describing the change along the characteristics of derivatives of order higher than the first and first integrals that ensure the reduction of the original equation to a system of ODEs. In the first and second methods, it is assumed that the independent variable in the system of equations of characteristics is the function — the solution to the original equation. In the case of the third method, the solution of the system of equations of characteristics is considered as a transition to new coordinates, and one of the new coordinates is the level surface of the solution.

The authors notes the experience of using the considered methods for a number of equations, in particular, for some equations of nonlinear acoustics [4], the convection-diffusion equation [5], the potential double wave equation in the hodograph plane [6], et al. In the report, these and some generalizing approaches are presented for the equation for the velocity potential in the axisymmetric case, the equation for the axisymmetric stationary laminar hydrodynamic boundary layer, the one-dimensional non-stationary filtration equation and the homogeneous Monge-Ampere equation.

REFERENCES

1. Sidorov A. F., Shapeev V. P. and Yanenko N. N. *Method of Dierential Constraints and its Applications in Gas Dynamics*, Nauka, Novosibirsk (1984). (in Russian)
2. Ovsiannikov L. V., *Group Analysis of Differential Equations*, Academic Press, New York (1982).
3. Clarkson P. A. and Kruskal M. D., *New similarity reductions of the Boussinesq equation*, J. Math. Phys., **30**(10), 2201–2213 (1989).
4. Rubina L. I. and Ulyanov O. N., *On solving certain nonlinear acoustics problems*, Acoust. Phys., **61**(5), 527–533 (2015).
5. Rubina L. I. and Ulyanov O. N., *On one approach to solving nonhomogeneous partial differential equations*, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, **27**(3), 355–364 (2017).
6. Rubina L. I. and Ulyanov O. N., *On double wave type flows*, Siberian Math. J., **60**(4), 673–684 (2019).

Normalization Property of Classes of Differential Equations and its Application in Group Analysis

O. O. Vaneeva

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine; vaneeva@imath.kiev.ua

An admissible transformation is a triple consisting of two fixed equations from a class and a transformation that links these equations. The set of admissible transformations considered with the standard operation of composition of transformations is also called the equivalence groupoid. It is important to study transformation properties of classes of differential equations, i.e. to describe explicitly their equivalence groupoids. Indeed, if two differential equations are connected by a nondegenerate point transformation, then associated objects like exact solutions, local conservation laws, and different kinds of symmetries of these equations are also related by this transformation. Such equations are called equivalent (or *similar* in terms of [1]). The knowledge of an exact solution for one of two equivalent equations allows one to construct the corresponding exact solution for the other equation using a point transformation connecting them. At the same time, nondegenerate point transformations appear to be a useful tool not only for finding exact solutions but also for exhaustive solution of group classification problems and study of integrability (see, e.g., [2] and references therein).

By Ovsiannikov [1], the equivalence group of a class consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary class elements that map any equation of this class to an equation from the same class, where the transformation components for the independent and the dependent variables are projectible on the space of these variables. After appearance of equivalence groups of other kinds the ones introduced by Ovsiannikov are called the usual equivalence groups. If transformation components for independent or dependent variables involve arbitrary elements, then the corresponding equivalence group is called the generalized equivalence group [3]. If target arbitrary elements appear to depend on source ones in a nonlocal way, then the corresponding equivalence group is called extended, whereas extended generalized equivalence groups possess both the aforementioned properties [4].

If any admissible transformation in a given class is induced by a transformation from its equivalence group (usual / generalized / extended / extended generalized), then this class is called *normalized* in the corresponding sense.

In this talk using illustrative examples we discuss how the normalization property is used in group analysis of differential equations, how it affects choosing methods of group classification to be applied and appropriate gaugings of arbitrary elements. These examples include classes of variable coefficient Kawahara equations and of (1+1)-dimensional nonlinear wave and elliptic equations.

The author gratefully acknowledges financial support by NAS of Ukraine within the project 0121U110543.

REFERENCES

1. Ovsianikov L. V., *Group Analysis of Differential Equations*, Academic Press, New York (1982).
2. Vaneeva O. O., Bihlo A. and Popovych R. O., *Generalization of the algebraic method of group classification with application to nonlinear wave and elliptic equations*, Commun. Nonlinear Sci. Numer. Simul., **91**, 105419 (2020).
3. Meleshko S. V., *Group classification of the equations of two-dimensional motions of a gas*, J. Appl. Math. Mech., **58**, 629–635 (1994).
4. Popovych R. O., Kunzinger M. and Eshraghi H., *Admissible transformations and normalized classes of nonlinear Schrödinger equations*, Acta Appl. Math., **109**, 315–359 (2010).

Differentiation of Similar Matrices

S. A. Vasyutkin[1], *A. P. Chupakhin*[2]

[1]*Novosibirsk State University, Novosibirsk, Russia*; s.vasyutkin@g.nsu.ru
[2]*Lavrentev Institute of Hydrodynamics, Siberian Branch of Russian Academy of Sciences, Novosibirsk State University, Novosibirsk, Russia*; alexander190513@gmail.com

The similarity transformation $T_g : a \longrightarrow gag^{-1}$ splits the set of square $n \times n$ matrices into equivalence classes. The Jordan form Λ_a of a matrix a is the canonical representative of each class [1]. The algebraic invariants of a matrix a are invariants of the similarity transformation T_g. They can be expressed in various ways, in particular, as the traces of the powers of a: $j_1 = tra, j_2 = tra^2, ..., j_n = tra^n$. An extensive literature is devoted to constructing bases for sets of matrices $a, b, ...$ and their properties. Let a matrix a depend smoothly on a parameter $t \in I \subseteq \mathbb{R}$, so that the derivatives $a_1 = \frac{da}{dt}, a_2 = \frac{d^2a}{dt^2}, ...$ are defined as matrices formed from the derivatives of the entries of the matrix. The eigenvalues and algebraic invariants of the matrices $a_1, a_2, ...$ are no longer expressed by simple formulas in terms of the algebraic invariants of the matrix a, except for the relation $\frac{d^k}{dt^k} tra = tr \frac{d^k a}{dt^k}$.

The question about the relation between the eigenvalues and algebraic invariants of the original matrix a and its derivatives $\frac{da}{dt}, \frac{d^2a}{dt^2} ...$ is of interest in itself and finds applications in continuum mechanics [1], in gas dynamics [2]. It is formulated and solved on the basis of the theory of differential invariants [3].

This paper provides formulas for calculating the algebraic invariants of derivatives of any order of the original matrix and gives the proof of their correctness with respect to the choice of the similarity matrix that reduces the original matrix a to its Jordan form Λ_a. We construct an invariant differentiation operator for the Lie group of continuous transformations $\{T_g\}$ implementing similarity. For 2×2 we discover a connection between the derivatives of a matrix and the differential analogue of the Clifford algebra [4].

REFERENCES

1. Godunov S. K. and Romenskii E. I., *Elements of Continuum Mechanics and Conservation Laws*, Kluwer Academic Plenum Publishers, New York (2003).
2. Sidorov A. F., Shapeev, V. P. and Yanenko N. N., *The Method of Differential Constraints and Its Applications in Gas Dynamics*, Nauka, Novosibirsk (1984).
3. Ovsjannikov L. V., Group Analysis of Differential Equations, English translation by W. F. Ames, Academic, New York, (1982).
4. Vasyutkin S. A. and Chupakhin A. P. *Differentiation of similar matrices*, Math Notes **109**, 302306 (2021).

On the Derivation of the Equations of Electrodynamics and Gravitation from the Least Action Principle and the Models of the Universe

V. V. Vedenyapin[1], *N. N. Fimin*[1], *V. M. Chechetkin*[1], *A. A. Russkov*[1], *M. Yu. Voronina*[1]

[1]*Keldysh Institute of Applied Mathematics of RAS, Moscow, Russian Federation*; vicveden@yahoo.com

In classical books (see [1]–[4]), equations for electromagnetic and gravitation fields are proposed without derivation of the right–hand sides. Here we give the derivation of the right-hand sides of the Maxwell and Einstein equations in the framework of the Vlasov–Maxwell–Einstein equations from the classical, but slightly more general principle of least action [5]–[11]. The resulting derivation of the Vlasov–type equations gives the Vlasov–Einstein equations different from those proposed earlier [12]–[15]. A method is proposed for the transition from kinetic equations to hydrodynamic consequences [5]–[8], as it was done earlier by A. A. Vlasov himself [4]. In the case of Hamiltonian mechanics, the transition to the Hamilton–Jacobi equation from the hydrodynamic consequences of the Liouville equation is possible, as was done already in quantum mechanics [16]. Thus, in the nonrelativistic case, we obtain the Milne–McCree solutions, a nonrelativistic analogue of the Friedmann–type solutions of the nonstationary evolution of the Universe.

REFERENCES

1. Fock V. A., *Theory of Space, Time and Gravitation*, London, Pergamon Press (1959).
2. Landau L. D., Lifshitz E. M., *The Classical Theory of Fields*, Oxford, Pergamon Press (1962).
3. Weinberg S., *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, New York, John Wiley and Sons, 1972.
4. Vlasov A.A., *Statistical Distribution Functions*, Moscow, Nauka (1966).
5. Vedenyapin V. V. and Negmatov M. A., *On the derivation and classification of equations of the Vlasov and MHD type. Lagrange's identity and Godunov's form*, Teor. Matem. Fiz., **170**(3), 468–480 (2012).
6. Vedenyapin V. V., Negmatov M. A., Fimin N. N. *Equations of the Vlasov and Liouville type, their microscopic, energetic and hydrodynamic consequences*, Izv. RAN., ser. Matem., **81**(3), 45–82 (2017).
7. Vedenyapin V. V., Negmatov M. A., *On the derivation and classification of equations of the Vlasov type and magnetohydrodynamics. Lagrange identity, Godunov form and critical mass*, Sovrem. Matem. Fundam. Napravl., **47**, 5–17 (2013).
8. Vedenyapin V. V., *Kinetic Boltzmann and Vlasov equations*, Moscow, Fizmatlit (2001).
9. Vedenyapin V. V., *Vlasov–Maxwell–Einstein equation*, Preprint of KIAM RAS 188, Moscow, KIAM RAS (2018)
10. Vedenyapin V. V., Fimin N. N., Chechetkin V. M. *The system of VlasovMaxwellEinstein-type equations and its nonrelativistic and weak relativistic limits*, Intern. J. Mod. Phys. D, **29**(1), 2050006-1 – 050006-23 (2020).
11. Vedenyapin V. V., Fimin N. N., Chechetkin V. M. *The properties of VlasovMaxwellEinstein equations and its applications to cosmological models*, Europ. Phys. J. Plus, **135**(5), 400-1 – 400-14 (2020).
12. Cercignani C., Kremer G. M. , *The Relativistic Boltzmann Equation: Theory and Applications*, Basel–Boston–Berlin, Birkhauser (2002).
13. Choquet–Bruhat Y., Damour T., *Introduction to General Relativity, Black Holes and Cosmology*, New York, Oxford University Press (2015).
14. Rein G., Rendall A. D., *Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data*, Commun. Math. Phys., **150**, 561–583 (1992).
15. Kandrup H. E., Morrison P. J., *Hamiltonian structure of the VlasovEinstein system and the problem of stability for spherical relativistic star clusters*, Ann. Phys., **225**, 114–166 (1993).
16. Madelung E., *Quantentheorie in hydrodynamischer form*, Zeit. Phys., **40**, 322–325 (1926).

Distinguished Limits and Drifts: between Nonuniqueness and Universality

V. A. Vladimirov

York University and Leeds University, United Kingdom; vv500@york.ac.uk

This paper deals with a version of the two-timing method which describes various 'slow' effects caused by externally imposed 'fast' oscillations. Such small oscillations are often called *vibrations* and the research area can be referred as *vibrodynamics*. The governing equations represent a generic system of first-order ODEs containing a prescribed oscillating velocity $\boldsymbol{u}$, given in a general form. Two basic small parameters stand in for the inverse frequency and the ratio of two time-scales; they appear in equations as regular perturbations. The proper connections between these parameters yield the *distinguished limits*, leading to the existence of closed systems of asymptotic equations. The aim of this paper is twofold: (i) to clarify (or to demystify) the choices of a slow variable, and (ii) to give a coherent exposition which is accessible for practical users in applied mathematics, sciences and engineering. We focus our study on the usually hidden aspects of the two-timing method such as the *uniqueness or multiplicity of distinguished limits* and *universal structures of averaged equations*. The main result is the demonstration that there are two (and only two) different distinguished limits. The explicit instruction for practically solving ODEs for different classes of $\boldsymbol{u}$ is presented. The key roles of drift velocity and the qualitatively new appearance of the linearized equations are discussed. To illustrate the broadness of our approach, two examples from mathematical biology are shown.

Key words: applied mathematics, differential equations, asymptotic methods, perturbation methods, two-timing method, vibrodynamics, distinguished limits, averaged equations, slow-time variable, universal structures, drift velocity.

Conformal Invariance of the 1-Point Statistics of the Zero-Isolines of 2d Scalar Fields in Inverse Turbulent Cascades

M. Wacławczyk[1], *V.N. Grebenev*[2], *M. Oberlack*[3]

[1]*Institute of Geophysics, Faculty of Physics, University of Warsaw Pasteura 7, 02-093 Warsaw, Poland*; marta.waclawczyk@igf.fuw.edu.pl

[2]*Federal Research Center for Information and Computational Technologies, Russian Academy of Sciences, Lavrentjev ave. 6, 630090 Novosibirsk, Russia*; vngrebenev@gmail.com

[3]*Department of Mechanical Engineering, Technische Universität Darmstadt Otto-Bernd-Str. 2, Darmstadt 64287, Germany*; oberlack@fdy.tu-darmstadt.de

This study concerns conformal invariance of certain statistics in a class of hydrodynamic models for scalar fields in $2d$. As it was discussed by Bernard et al. [1], there exist numerical evidence that the zero-vorticity isolines $\mathbf{x}(l,t)$ for the $2d$ Euler equation with an external force and a uniform friction belong to the class of conformally invariant random curves. Based on this evidence, the conformal invariance was formally proven in Ref. [2] by a Lie group analysis for the 1-point probability density function governed by the inviscid Lundgren-Monin-Novikov equations for $2d$ vorticity fields subject to the zero-vorticity constraint $\omega = 0$. Therein, no external forcing was considered.

In this work we consider the first equation from the Lundgren-Monin-Novikov chain for $2d$ scalar fields ϕ under Gaussian white-in-time forcing and large-scale friction. With this, the flow can be kept in a statistically steady state and the analysis is performed for the stationary Lundgren-Monin-Novikov equation. We show that the conformal invariance can be retained in the presence of large-scale friction and forcing under the restriction $\phi = 0$, however, it is broken if the viscous term is included into the equation. Specifically, for the inviscid case we prove the conformal invariance of the 1-point statistics of the zero-isolines $\mathbf{x}(l,t)$ of a scalar field, i.e. the conformal invariance of the probability $f_1(\mathbf{x}(l),\phi)d\phi$ that a random curve $\mathbf{x}(l,t)$ passes through the point $\mathbf{x}$ with $\phi = 0$ for $l = l_1$.

We show an example, where the proposed transformations represent a change from probability density functions describing homogeneous fields to the ones that describe non-homogeneous turbulence. Possible implications of this result are discussed.

REFERENCES

1. Bernard D., Boffetta G., Celani A., and Falkovich G., *Conformal invariance in two-dimensional turbulence*, Nature Physics, **2**, 124–128 (2006).
2. V.N. Grebenev, M. Wacławczyk, and M. Oberlack, *Conformal invariance of the zero-vorticity Lagrangian path in 2D turbulence*, J. Phys. A: Math. Theor. **50**, 335501 (2019).

Conservation Laws and Action Principles for CGL Plasmas

G. M. Webb

CSPAR, The University of Alabama in Huntsville, Huntsville AL35805, USA;
gmw0002@uah.edu

This paper uses the Lagrangian action formulation of the Chew-Goldberger-Low (CGL) equations of plasma physics, in which the plasma pressure is anisotropic, with components $p_{\|}$ and $p_{\perp}$ parallel and perpendicular to the magnetic field $\mathbf{B}$, where $p_{\|}$ and $p_{\perp}$ satisfy the double adiabatic equations which are related to the first and second adiabatic invariants ([1],[2], [4]). Euler-Poincaré and Lagrangian formulations of the equations are developed, which depend on the internal energy per unit mass of the plasma. Noether's theorem is used in conjunction with the Lagrangian variational principle to derive conservation laws for the equations, using the approach ([3],[5]). Conservation laws corresponding to the 10 parameter Galileian group are obtained. These symmetries give rise to the energy (time translation invariance), momentum (space translations), center of mass conservation law (Galileian boosts) and the angular momentum conservation laws (rotational symmetries about an axis of rotation). A non-local cross helicity conservation law associated with a fluid relabelling symmetry is also obtained. Hamiltonian Poisson bracket formulations of the equations are discussed.

The author was supported in part by NASA grant 80NSSC19K0075. GMW acknowledges stimulating discussions with Darryl Holm, Stephen Anco, Sergey Meleshko, Peter Hunana and Gary Zank.

REFERENCES

1. Chew G. L., Goldberger M. L. and Low F. E., *The Boltzmann equation and one-fluid hydromagnetic equations in the absence of particle collisions*, Proc. Roy. Soc. London, Series A, **236**, 112–118 (1956).
2. Newcomb W. A., *Lagrangian and Hamiltonian Methods in magnetohydrodynamics*, Nucl. Fusion Suppl., Part 2, 451–463 (1962).
3. Webb G. M. and Anco S. C., *Conservation laws in magnetohydrodynamics and fluid dynamics: Lagrangian Approach*, AIP Proc. Conf., **2153**, (1):020024, 8 pp., in Modern Treatment of Symmetries, Differential Equations and Applications (Symmetry 2019).
4. Hunana P. et.al., *Lecture Notes: An introductory guide to fluid models with anisotropic temperatures, Part 1, CGL description and collisionless fluid hierarchies*, J, Plasma Phys., **85**, 205850602, 182pp. (2019).
5. Webb G., *Magnetohydrodynamics and Fluid Dynamics: Action Principles and Conservation Laws*, Lecture Notes in Physics, **946**, Springer, 301pp. (2018).

Author Index

Aibinu M. O.., 1
Alcántara-Álvila F., 51
Alekseev G. V., 2
Algazin S. D., 3
Almusawa H., 28, 29
Anco S. C., 4
Andreev V. K., 5
Anikin A. Yu., 6
Askhabov S. N., 7
Assylbekuly A., 16

Baenova G. M., 8
Baghaturia G., 58
Bellani G., 51
Bila N., 9
Bobrovskiy V. S., 10
Bogdanov, A. N., 11
Brizitskii R. V., 2
Bulatov M. V., 12

Cada M., 18, 19
Campoamor-Stursberg R., 13
Chechetkin V. M., 77
Cherepanov A. N., 33
Cherevko A. A., 42, 65
Chernykh G. G., 14, 25
Chesnokov A. A., 15
Cheviakov A., 16, 70
Chupakhin A. P., 76
Conte R., 17

de la Cruz A., 18, 19
Demenkov A. G, 25
Diaz-Chang T., 18, 19
Dimakis N., 20
Dobrokhotov, S. Yu., 21
Dorodnitsyn V. A., 34
Dryuma V. S., 22
Duba C. T., 30
Dutykh D., 16
Dyachenko A. I., 23
Dyachenko S. A., 23

Fimin N. N., 77
Fomina A. V., 14

Gavrilyuk S., 24
Grebenev V. N., 25
Grebenev V. N., 79
Grigoriev Yu. N., 26
Grundland A. M., 17

Habibullin I. T., 27
Halder A. K., 28–30
Holba P., 31
Hollmann P., 51

Hoyas S., 51
Hultmark M., 51

Il'in, V. P., 32
Isaev V. I., 33

Kaptsov E. I., 34
Kaptsov O. V., 35
Karabut E. A., 36
Kara A. H., 50
Kazakov A. L., 37
Klewicki J., 51
Klingenberg D., 51
Kovtunenko V. A., 38
Kraheberger S., 51
Kudryashov N. A., 39
Kukushkin D. E., 41
Kulikov E. K., 40

Laux J., 51
Leach P. G. L., 28–30
Lee M.-G., 37
Lemeshkova E. N., 5
Lempert A. A., 37
Liang Ch., 18, 19
Liapidevskii V. Yu., 15

Mahdipour–Sh. A., 48
Makarenko N. I., 41, 42
Makarov A. A., 40
Maltseva J. L., 42
Manganaro N., 43
Meleshko S. V., 26, 34, 44, 49
Millionshchikov D., 45
Mirzaokhmedov M. M., 35
Mitsopoulos, A., 46
Moshkin N. P., 14
Moyo S., 1
Muriel C., 47

Nadjafikhah M., 48
Nakpim W., 49
Naseer S., 50
Nazaikinskii, V. E., 21

Oberlack M., 51, 79
Olver, P. J., 52
Orhan Ö., 53, 59
Ostapenko V. V., 65

Paliathanasis A., 28, 29
Paliathanasis, A., 54
Palymskiy I. B., 55
Palymskiy V. I., 55
Pavlov M. V., 56
Pelinovsky E., 57

Peradzynski Z., 58
Phuong T. D., 67
Pinar Z., 59
Pistora J., 18, 19
Pukhnachev V. P., 60
Pınar Z., 53

Raza A., 50
Rogalev A. N., 61
Romero, J. L., 47
Rozanova O. S., 62
Rubina L. I., 74
Ruiz A., 47
Russkov A. A., 77
Rykhlov V. V., 6

Seesanea A., 63
Selivanov I. A., 3
Sergyeyev A., 64
Shapeev V. P., 33, 41
Sharifullina T. S., 65
Shyue K.-M. , 24
Sinitsyn A. V., 10
Siraeva D. T., 66
Siriwat P., 26, 44
Smits A., 51
Solovarova L. S., 12, 67
Stepanyants Y. A., 68
Sukhinin S. V., 8

Talamelli A., 51
Talipova T., 57
Talyshev A. A., 69
Tarayrah M. R., 70
Thakur S. C., 1
TKachev D. L., 71
Tolchennikov, A. A. , 21
Torres R., 72
Tsamparlis, M., 46
Tsarev S. P., 73

Ulyanov O. N., 74

Vallikivi M., 51
Vaneeva O. O., 75
Vasyutkin S. A., 76
Vedenyapin V. V., 77
Vladimirov, V. A., 78
Voronina M. Yu., 77

Wacławczyk M., 79
Webb, G. M., 80

Zakharov V. E., 23
Zaman F. D., 50
Zhumadillayeva A. K., 8
Zhuravleva E. N., 36
Zimmerman S., 51