Symmetry 2019 @ SUT, Thailand

Conference Program
and
Book of Abstracts

(Current as of January 7, 2019)

Modern Treatment of Symmetries,
Differential Equations and Applications
(Symmetry 2019)

14 – 18 January, 2019

Suranaree University of Technology
Nakhon Ratchasima, Thailand

Conference Program

Monday, 14 January 2019

08.00 – 09.00	Registration
09.00 – 09.20	Opening Ceremony
09.20 – 09.50	The Life of Academician Lev Vasilievich Ovsyannikov
09.50 – 10.00	Group Photo
	Coffee Break
	Chair: Aptekarev A. I.
10.15 – 10.45	**Golovin, S. V.** TBA
11.15 – 11.45	**Olver P. J.** Differential invariants, moving frames, equivalence and symmetry, image processing, and the reassembly of broken objects
10.45 – 11.15	**Makarenko N. I. and Makridin Z. V.** Bifurcation of periodic solutions to nonlinear dispersive systems with symmetries
11.45 – 12.15	**Meleshko, S.V.** Relationships between the group analysis method and the method of differential constraints
	Lunch in Academic Building C2
	Chair: Dorodnitsyn V. A.
13.30 – 14.00	**Aptekarev A. I.** Jacobi matrices on trees and discrete integrable systems
14.00 – 14.30	**Afendikov A.** Symmetries, cosymmetries and theory of bifurcations without parameter in some hydrodynamic problems
14.30 – 15.00	**Shakiban C. and Grim A. M.** Applications of signatures curves to characterize melanomas and moles
	Coffee Break
	Chair: Kovalev V. F.
15.15 – 15.45	**Mkhize T. G., Govinder K., Moyo S. and Meleshko S.V.** Linear system of two second-order stochastic ordinary differential equations
15.45 – 16.15	**Chong K. Y. and O'Hara J. G.** Lie symmetry analysis of a fractional Black-Scholes equation
16.15 – 16.45	**Suriyawichitseranee A., Grigoriev Yu. N., Meleshko S. V.** Group analysis and exact solutions of the spatially homogeneous and isotropic Boltzmann equation with a source term
	Dinner in Academic Building C2

Tuesday, 15 January 2019

	Chair: Leach P. G. L.
09.00 – 09.30	**Ruggeri T.** Galilean invariance and entropy principle for a system of balance laws of mixture type
09.30 – 10.00	**Kovtunenko V. A.** Entropy method for generalized Poisson–Nernst–Planck equations
10.00 – 10.30	**Wessels E. J. H.** On the metric tensor in the external gravitational field of an isolated, spherically symmetric, non-rotating massive object
Coffee Break	
	Chair: Afendikov A.
10.45 – 11.15	**Muriel C., Romero, J. L. and Ruiz, A.** Integration methods
11.15 – 11.45	**Evnin, O.** Weakly nonlinear dynamics of strongly resonant systems
11.45 – 12.15	**Makridin, Z.** Multi-Dimensional Conservation Laws for Integrable Systems
Lunch in Academic Building C2	
	Chair: Ruggeri T.
13.30 – 14.00	**Webb G. M. and Anco S. C.** Conservation laws in magnetohydrodynamics, Lagrangian, Clebsch and multi-symplectic spproaches
14.00 – 14.30	**Kaptsov E. I. and Meleshko S. V.** One-dimensional continuum mechanics with a Lagrangian of a special form
14.30 – 15.00	**Kaewmanee C. and Meleshko S. V.** Conservation laws of one-dimensional equations of fluids in Lagrangian coordinates
Coffee Break	
	Chair: Webb G. M.
15.15 – 15.45	**Voraka P., Kaewmanee C. and Meleshko S. V.** Symmetries of the shallow water equations in the Boussinesq approximation
15.45 – 16.15	**Ruiz A. and Muriel C.** Use of a solvable pair of variational $\mathcal{C}^\infty$-symmetries to reduce the order of Euler–Lagrange equations
16.15 – 16.45	**AlKindi F. M. and Ziad M.** Solutions of systems of ordinary differential equations using invariants of symmetry groups
Dinner banquet off-site *Return to the hotel no later than 22.00*	

Wednesday, 16 January 2019

	Excursion Day : Program TBA

Thursday, 17 January 2019

	Chair: Moyo, S.
09.00 – 09.30	**Dorodnitsyn V. A., Kozlov R., Meleshko S. V. and Winternitz P.** Lie group classification and integration of delay ordinary differential equations
09.30 – 10.00	**Athorne C.** Equivariance in the theory of higher genus ℘-functions
10.00 – 10.30	**Takemura K.** Heun's differential equation and its q-deformation
	Coffee Break
	Chair: Chaiyasena, P.
10.45 – 11.15	**Aksenov A. V. and Druzhkov K. P.** The two-dimensional shallow water system over a rough bottom. Conservation laws
11.15 – 11.45	**Aksenov A. V. and Druzhkov K. P.** The two-dimensional shallow water system over a rough bottom. Group classification
11.45 – 12.15	**Kudryashov, N.** Higher order Painlevé equations and some of their properties
	Lunch in Academic Building C2
	Chair: Muriel, C.
13.30 – 14.00	**Paliathanasis, A.** Symmetry analysis and inflation
14.00 – 14.30	**Halder A. K. and Leach P. G. L.** Lie point symmetries and conservation laws of a system of time fractional partial differential equation
14.30 – 15.00	**Panov A. V.** Some submodels of gas suspension equations with respect to three dimensional subalgebras
	Coffee Break
	Chair: Athorne C.
15.15 – 15.45	**Chaiyasena A. P.** Lie approach to flow down a nonuniform tube
15.45 – 16.15	**Tantanuch J.** Equation of a Rayleigh noise reduction model for medical ultrasound imaging: symmetry classification, conservation laws and invariant solutions
16.15 – 16.45	**Namngam, K. and Schulz E.** Admissible vectors for a class of subgroups of the symplectic group
	Dinner at own leisure

Friday, 18 January 2019

	Chair: Olver P. J.
09.00 – 09.30	In Memoriam Nail H. Ibragimov
09.30 – 10.00	**Charalambous K., Halder A. K. and Leach P. G. L.** Analysis of the Kaup-Kupershmidt equation
10.00 – 10.30	**Kovalev V. F.** Approximate RG-symmetry and invariant solutions to the relativistic self-focusing problem
Coffee Break	
	Chair: Makarenko. N.
10.45 – 11.15	**Talyshev A. A.** On models of dynamics material points that are invariant with respect to the Poincaré group
11.15 – 11.45	**Svirshchevskii S. R.** Exact solutions for nonlinear heat equation with maximal Symmetry algebra
11.45 – 12.00	Closing Ceremony
Lunch in Academic Building C2	

List of Abstracts

Symmetries, Cosymmetries and Theory of Bifurcations without Parameter in Some Hydrodynamic Problems

A. Afendikov

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In the early 1990s V.Yudovich introduced the notion of cosymmetry. He discovered that in this case steady states are generically non-isolated and investigated some bifurcation problems. Recently several hydrodynamic problems in unbounded domains where in a vicinity of the instability threshold the dynamics is governed by the generalized Cahn-Hilliard equation were considered. For the time independent solutions of this equation Bogdanov-Takens bifurcation without parameter in the 3-dimensional reversible system with a line of equilibria were recovered. This line of equilibria is neither induced by symmetries, nor by first integrals. At isolated points, normal hyperbolicity of the line fails due to a transverse double eigenvalue zero. The bi-reversible problem and its small perturbation with only one symmetry left were studied in [1, 2]. Our aim is to relate Yudovich theory to problems with symmetry and to discuss hydrodynamic problems, where the reversibility breaking perturbation can't be considered as small.

This is a joint work with Sergey Denisov and Maxim Yattselev.

Supported by RSCF 17-71-30014.

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2. A. Afendikov, B. Fiedler, and S. Liebscher, *Plane Kolmogorov flows and Takens-Bogdanov bifurcation without parameters: The singly reversible case*, Asymptotic Analysis, **72**(1–2), p.31–76, 2011.

The Two-Dimensional Shallow Water System over a Rough Bottom. Group Classification

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The system of equations of two-dimensional shallow water over a rough bottom [1] is considered. An overdetermined system of equations for finding the allowed symmetries [2] is obtained. The consistency of this overdetermined system of equations is investigated. A general form of the solution of this system is obtained. The kernel of the symmetry operators is found. Cases are presented where kernel extensions of symmetry operators exist. The corresponding classifying equations are given. The results of the group classification have indicated that the system of equations of two-dimensional shallow water over a rough bottom cannot be linearized by point change of variables in contrast to the system of equations of one-dimensional shallow water in the cases of horizontal and inclined bottom profiles [3].

The authors gratefully acknowledge financial support by RFBR grant No. 18-01-00890.

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2. Ovsiannikov L. V., *Group Analysis of Differential Equations*, Academic Press, New York (1982).
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The Two-Dimensional Shallow Water System over a Rough Bottom. Conservation Laws

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A system of equations of two- dimensional shallow water above the rough bottom [1] is considered. An overdetermined system of equations for determining the functions forming the conservation laws of the system of shallow water equations is obtained. The general form of the solution of the overdetermined system is found. The general classification equation is given. The system of equations of two-dimensional shallow water above the rough bottom for any profile of the bottom is shown to have no more than the nine-dimensional space of the hydrodynamic conservation laws. The new conservation law, supplementary to the basic conservation law, is obtained (as in one-dimensional case [2]). All of the hydrodynamic conservation laws have found for all possible bottom profiles.

The authors gratefully acknowledge financial support by RFBR grant No. 18-01-00890.

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2. Aksenov A. V. and Druzhkov K. P., *Conservation laws and symmetries of the shallow water system above rough bottom.*, J. Phys.: Conf. Ser., **722**, 1-7 (2016).

Solutions of Systems of Ordinary Differential Equations Using Invariants of Lie Point Symmetry Groups

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For a system of one or more n-th order ordinary differential equations (ODEs)

$$F_i(t, \mathbf{x}, \mathbf{x}', ..., \mathbf{x}^n) = 0, \tag{1}$$

admitting the one parameter Lie group of transformations with infinitesimal generator

$$X = \xi(t, \mathbf{x})\frac{\partial}{\partial t} + \eta_i(t, \mathbf{x})\frac{\partial}{\partial x_i}, \ i = 1, ..., k, \tag{2}$$

the invariant solution is a function $\phi(\mathbf{x})$ that is an invariant curve of (2) and solves the system (1). In this paper, Bluman's theorem [1] of invariant solutions of a single ODE is extended for systems of ODEs. The statement of the theorem follows as:

Theorem 1. *Suppose that the system of ODEs* (1) *admits a one parameter Lie group of transformations with infinitesimal generator* (2) *in Domain* $\boldsymbol{D} \subset \boldsymbol{R^{k+1}}$. *Assume that* $\xi(t, \mathbf{x}) \neq 0$ *in* $\boldsymbol{D}$. *Let*

$$\psi_i(t, \mathbf{x}) = \frac{\eta_i(t, \mathbf{x})}{\xi(t, \mathbf{x})}, \ \ Y = \frac{\partial}{\partial t} + \psi_i(t, \mathbf{x})\frac{\partial}{\partial x_i} = \frac{1}{\xi(t, \mathbf{x})}X,$$

and

$$Q_i(t, \mathbf{x}) = F_i(t, \mathbf{x}, \psi_1, ..., \psi_k, Y\psi_1 ..., Y\psi_k, ..., Y^{n-1}\psi_1, ..., Y^{n-1}\psi_k). \tag{3}$$

1. *If any of* $Q_i(t, \mathbf{x}) = 0$ *has no solution in* $\boldsymbol{D}$, *then the system* (1) *has no invariant solutions related to its invariance under* (2).
2. *If* $Q_i(t, \mathbf{x}) \equiv 0 \ \forall i$, *in* $\boldsymbol{D}$, *then each invariant curve of* (2) *is an invariant solution of the system* (1).
3. *If* $Q_i(t, \mathbf{x}) \not\equiv 0$ *but* $Q_i(t, \mathbf{x}) = 0$ *define curves in* $\boldsymbol{D}$, *then these curves define invariant solutions for the system* (1) *in* $\boldsymbol{D}$.

We will prove this theorem and discuss the case when $\xi = 0$ followed by some examples.

Moreover, if the system (1) admits a Lie group G_r of symmetry generators

$$X_j = \xi_j(t, \mathbf{x})\frac{\partial}{\partial t} + \eta_{ji}(t, \mathbf{x})\frac{\partial}{\partial x_i}, \ \ j = 1, ..., r, \tag{4}$$

then the differential invariants of this group will be used to find first integrals of the system. A few examples for system of ODEs will be provided.

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Jacobi Matrices on Trees and Discrete Integrable Systems

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We consider a discrete integrable system which produces coefficients of the recurrence relations for the multiple orthogonal polynomials on the lattice $(\mathbb{Z}^+)^d$. We use these relations to construct a self-adjoint operator (Jacobi matrix) on the tree. This tree is a homogenous infinite rooted tree with homogeneity degree which is equal to $d+1$ (i.e., each vertex has one "parent" (incoming) edge and d "children" (outgoing) edges). The case $d=1$ gives the polynomials orthogonal on the real line, the tree becomes $\mathbb{Z}^+$, and the Jacobi matrix is the standard three-diagonal matrix.

We shall discuss multiple orthogonal polynomials approach to the spectral theory of multidimensional discrete Schrödinger operators and corresponding discrete integrable systems.

This is a joint work with Sergey Denisov and Maxim Yattselev.

Equivariance in the Theory of Higher Genus $\wp$-Functions

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We will discuss the way that simple transformations of algebraic curves give rise to transformations of the differential equations satisfied by meromorphic, rational functions on the curve in three contexts.

Consider firstly a family of genus g hyperelliptic curves $y^2 = a(x)$, where $a(x)$ is a polynomial of degree $2g+1$ or $2g$. Rational linear maps $x \to \frac{ax+b}{cx+d}$, $y \to \frac{y}{cx+d}$ permute members of this family and induce g-dimensional fundamental representations of $SL_2(\mathbb{C})$ on the vector space of holomorphic differentials on the curve. The Abel map constructs a Jacobian variety of (complex) dimension g where the generalised (multiperiodic) $\wp$-functions live. The representation carried by the differentials gives a symmetry action on the set of diferential equations satisfied by the $\wp$-functions, generalising the simple genus one case (the Weirstraß wp-function). This allows one to derive and represent these differential equations in a straightforward manner by looking at various highest weight relations [1]. The Hirota derivative plays an important part in this theory [2]

Secondly, for a general compact, non-singular Riemann surface, the the Riemann-Roch theorem dictates that rational functions on the curve must satisfy relations (of which any particular model of the curve is an example). We observe that these relations can be classified as modules under a creation/annihilation map on divisors: $n.P + m.Q \to (n+1).P + (m-1).Q$, P and Q being points on the Riemann surface. This allows us to construct an equivariant free resolution on the pole-graded ring of meromorphic functions by which we can describe an exhaustive set of relations. Similar ideas may be extended to relations on the Jacobian of the sort discussed in the paragraph above [3].

Finally, by extending the transformation theory to differentials of the second kind one can build an equivariant Hamiltonian treatment of the generalized σ-function via the Gauss-Manin connection.

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Lie Approach to Flow Down a Nonuniform Tube

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The one dimensional formulation for flow in a tube of given cross sectional area $A(x, p)$ depending on position and pressure. We also assume the case where the sound speed is given by $\alpha^2 = \gamma p/\rho$. The Lie group generators of the equations are determined, and the optimal subalgebras are computed. Finally, solutions of the classification equation yielding particular forms of A corresponding to each optimal subalgebra.

Lie Symmetry Analysis of a Fractional Black-Scholes Equation

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The well-known Black-Scholes equation has been studied in many different perspectives since it was introduced by Black and Scholes in 1973 [1]. Its influence in the option market is huge and remarkable. The Black-Scholes equation is a partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0 \tag{1}$$

where $u =$ the price of a derivative, $x =$ the price of the stock, $\sigma =$ the volatility of the underlying asset, $r =$ the risk-free interest rate continuously compounded and $t =$ time in years.

In 2000, Walter Wyss looked into a fractional version of the Black-Scholes equation for the first time [2]. He gave a complete solution of the fractional Black-Scholes equation by using the Green's function. His work was complicated and comprehensive.

We used the Lie symmetry analysis to study the fractional verion of the equation (1), namely

$$D_t^\alpha u + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0 \tag{2}$$

where $\alpha \in$ (0,1), $D_t^\alpha u = \partial^\alpha u \partial t^\alpha$, $u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

Using the prolongation formula suggested by Gazizov [3] and simplified by Huang and Zhdanov [4], we gave the symmetry group of the equation (2). Finally, we gave an example of exact solution of the fractional Black-Scholes equation.

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Lie Group Classification and Integration of Delay Ordinary Differential Equations

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Lie group classification of first-order and second-order delay ordinary differential equations is presented. A delay ordinary differential system (DODS) is a delay ordinary differential equation (DODEs) accompanied by a delay relation, i.e. an equation which describes a delay parameter. A subset of such systems (delay ordinary differential systems or DODSs), which consists of linear DODEs and solution-independent delay relations, have infinite-dimensional symmetry algebra - as do nonlinear ones that are linearizable by an invertible transformation of variables. Genuinely nonlinear first order DODSs have symmetry algebras of dimension n, . It is shown how exact analytical solutions of invariant DODSs can be obtained using symmetry reduction. A Noether-type identities is obtained and used for constructing first integrals for DODS. This work was partially supported by Russian Fund for Base Research 18-01-00890.

This work was partially supported by Russian Fund for Base Research 18-01-00890.

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Weakly Nonlinear Dynamics of Strongly Resonant Systems

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A number of PDEs of mathematical physics (e.g., nonlinear Schroedinger equations in harmonic potentials) display the feature of having strongly resonant linearized perturbations (differences of any two normal mode frequencies are integer in appropriate units). In such situations, arbitrarily small nonlinearities of order g may produce arbitrarily large effects if one waits for long times of order 1/g. The "resonant approximation" captures leading effects of this sort, and results in a simplified dynamical system accurately approximating the weakly nonlinear dynamics. This resonant system is often much more structured and tractable than the original PDE. I will present a few explicit examples where such simplifications emerge.

Lie Point Symmetries and Conservation Laws of a System of Time Fractional Partial Differential Equations

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We discuss the Lie point symmetries and possible reductions of a system of time fractional partial differential equations. We propose the singularity analysis to look for the integrability of the reduced fractional ordinary differential equations. The conservation laws are mentioned explicitly.

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Conservation Laws of One-Dimensional Equations of Fluids in Lagrangian Coordinates

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One-dimensional motion of fluids in Lagrangian coordinates are considered in this study. The observation that the equations of fluids with internal inertia in Lagrangian coordinates have the form of an Euler-Lagrange equation with a natural Lagrangian allows us to apply Noether's theorem for constructing conservation laws for the equations of internal inertia. The complete group classification of one-dimensional of the equations of the gas dynamics type in Lagrangian coordinates is obtained. Using Noether's theorem, conservation laws in Lagrangian coordinates are constructed. For the hyperbolic shallow water equations and the Green-Naghdi equations new conservation laws are found.

This work was supported by Naresuan University Research Fund through grant R2560C145.

Analysis of the One-dimensional Euler-Lagrange Equation of Continuum Mechanics with a Lagrangian of a Special Form

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Flows of one-dimensional continuum in Lagrangian coordinates are studied in this presentation. Equations describing these flows are reduced to a single Euler-Lagrange equation which contains two undefined functions. Particular choices of the undefined functions correspond to isentropic flows of an ideal gas or different forms of the hyperbolic shallow water equations. A complete group classification of the equation with respect to these functions is performed.

Using Noether's theorem, all conservation laws are obtained, and their analogs in Eulerian coordinates are given.

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Approximate RG-Symmetry and Invariant Solutions to the Relativistic Self-Focusing Problem

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Approximate transformation groups are discussed in application to a mathematical model based on the nonlinear Shrödinger equation that describes the formation of a self-focusing structure of a laser beam in a plasma with relativistic nonlinearity. Nonlinear effects in plasma are due to the relativistic nonlinearity of the electron mass and the nonlinear deformation of the electron density. The application of the renormalization-group symmetry method makes it possible to determine approximate transformation group and construct group-invariant solutions for arbitrary chosen boundary conditions that describe smooth radial beam intensity distribution at the plasma boundary. The case of a laser beam with a Gaussian radial intensity distribution at the boundary is considered in detail. Different self-focused waveguide propagation modes with respect to controlling laser-plasma parameters are studied. The proposed theory specifies the domains and their boundaries on the plane of the controlling parameters where three distinct types of solutions, namely self-trapping, self-focusing on the axis, and tubular self-focusing solutions occur. The waveguide propagation modes are illustarted by spatial distributions of the laser beam intensity and the electron density at different distancies from the plasma boundary both for self-trapping and for self-focusing solutions.

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Entropy Method for Generalized Poisson–Nernst–Planck Equations

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To describe electro-kinetic transport phenomena occurring for micro-structures in many physical, chemical, and biological applications, a proper mathematical model adhering to the law of conservation of mass is suggested following the approach [1, 4]. The reference two-phase medium composed or pore and particle parts is described by nonlinear Poisson–Nernst–Planck (PNP) equations for concentrations of charged species and overall electrostatic potential. For physical consistency, they are generalized with entropy variables associating the pressure and quasi-Fermi electrochemical potentials.

Based on a suitable free energy, in [10] a variational principle is established within the Gibbs simplex, thus preserving the mass balance and positive species concentrations. The generalized PNP problem takes into account for nonlinear interface reactions which are of primary importance in applications. We provided the problem by rigorous asymptotic analysis in [2, 3], and by a-priori energy and entropy estimates in [8, 9]. Based on the entropy variables and following the formalism given in [5, 7, 6], further the PNP system is endowed with the structure of a gradient flow.

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Higher Order Painlevé Equations and Some of Their Properties

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The Painlevé equations were discovered more than one hundred ago as equations with solutions without critical movable points on the complex plane [1, 2]. For a long time, it was believed that these equations have interesting properties, but do not have or any applications in the description of physical or any other processes. Relations to Painlevé equations changed in the seventies of the last century after the discovery of the inverse scattering problem method for solving the Cauchy problem for nonlinear evolutionary solutions with soliton solutions. In paper [3] was found that invariant solutions of many nonlinear evolution equations are expressed through the solution of the Painlevé equations. In addition, almost at the same time, nonlinear evolution equations with soliton solutions appeared in the description of many other physical processes. As a result, there was a great interest in the study of Painlevé equations. At this stage, many remarkable properties characteristic of the Painlevé equations were discovered. In the eighties of the last century, it was demonstrated that the problem of solving the Cauchy problem for Painlevé equations by means of the Inverse Monodromy Transform method.

It was found that five of the six equations have the Backlund transformations, which allow us to find a set of rational solutions for certain values of the parameters of equations. The values of the parameters were found when the General solution of Painleve equation except the first one degenerate into the classical transcendental functions.

In the nineties of the last century, it was strictly proved that the general solutions of all six Painlevé equations are non-classical functions, which is a consequence of the fact that these equations do not have the first integrals in polynomial form.

For a long time, mathematicians have been unable to specify other equations with solutions similar to Painlevé functions. Paul Painlevé was convinced himself that there are only six irreducible second-order equations with non-classical functions. However he believed that other equations having general solutions expressed through essentially transcendental functions can be found among the fourth-order equations. The answer to this question was not until 1997 when the first and second Painlevé hierarchies were introduced in work [4]. Since then, this direction has become firmly established in the periodic scientific literature in which many new results related to this direction have been presented.

In this talk we consider higher-order Painleve equations with general solutions in the form of non-classical functions. The basic focus is on the well-known first and second Painleve hierarchies and their relation to the KDV equation written via self-similar variables. Some new hierarchies with properties similar to the Painlevé hierarchies are presented. It was shown that these hierarchies are reduced to the first and second Painleve hierarchies by means of non-local transformations. We give examples of the linear system of equations that are associated with these hierarchies. These Lax pairs for Painleve hierarchies can be used for solving of solving Cauchy problems of the hierarchies under consideration. Thus, the list of nonlinear differential equations whose general solutions are expressed in terms of non-classical functions are extended.

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Analysis of the Kaup-Kupershmidt Equation

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We study the Kaup-Kupershmidt Eqation in both its $1+1$ ans $1+2$ forms from the point of view of the possession of Lie point symmetries and the singularity analysis of the reduced scalar ordinary forms obtained by a consideration of the travelling-wave solution. In the case of the $1+2$ equation it is necessary to deal with a sixth-order equation to enable the removal of two integral terms.

In both cases the solution of the corresponding travelling-wave equation is shown to pass the Painlevé Analysis with both a Right Painlevé Series and a solution over an annulus.

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Bifurcation of Periodic Solutions to Nonlinear Dispersive Systems with Symmetries

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Bifurcations of periodic solutions in autonomous nonlinear systems of weakly coupled equations are studied. A comparative analysis is carried out between the mechanisms of LyapunovSchmidt reduction of bifurcation equations for solutions close to harmonic oscillations and cnoidal waves. The reduction is related with symmetry and cosymmetry properties of the original system. Sufficient conditions for the branching of orbits of solutions are formulated in terms of the Pontryagin functional depending on perturbing terms.

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Multi-Dimensional Conservation Laws for Integrable Systems

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We introduce and investigate a new phenomenon in the Theory of Integrable Systems — the concept of multi-dimensional conservation laws for two- and three-dimensional integrable systems. Existence of infinitely many local two-dimensional conservation laws is a well-known property of two-dimensional integrable systems. We show that pairs of commuting two-dimensional integrable systems possess infinitely many three-dimensional conservation laws.

Examples: the Benney hydrodynamic chain, the Korteweg de Vries equation.

Simultaneously three-dimensional integrable systems (like the Kadomtsev — Petviashvili equation) have infinitely many three-dimensional conservation laws. The method is based on introducing of auxiliary quasi-local variables (moments). It allows us to construct infinitely many multi-dimensional conservation laws depending on an arbitrary number of independent variables, which is higher time variable for commuting flows of each integrable hierarchy.

We illustrate our approach considering the dispersionless limit of the Kadomtsev — Petviashvili equation and the Mikhalëv equation.

Relationships Between the Group Analysis Method and the Method of Differential Constraints

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The presentation gives examples of mutual relationships between two methods. As a first example, generalized simple waves of the one-dimensional gas dynamics equations are considered. It is shown that the solutions, which were obtained by the method of differential constraints, can also be derived as partially invariant solutions, but in the extended space. As another example, new results on the construction of conservation laws of equations of a polytropic gas are presented. The group classification of one-dimensional equations of a polytropic gas in Lagrangian coordinates separates out three types of the entropy, whereas these relations in Eulerian coordinates become differential constraints. Adding this equation leads to new conservation laws in Eulerian coordinates. A slightly different situation occurs in constructing conservation laws for two-dimensional equations of a polytropic gas. However, as in the one-dimensional case, the group classification in Lagrangian coordinates leads to new conservation laws in Eulerian coordinates with the necessity to add an additional equation to the original gas dynamics equations.

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Linear System of Two Second-Order Stochastic Ordinary Differential Equations

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indexMoyo S.

Firstly, we give a new treatment for the linearization of two second-order stochastic ordinary differential equations. We provide the necessary and sufficient conditions for the linearization of these differential equations. The linearization criteria are given in terms of coefficients of the system. We further, consider the underlying group theoretic properties of a system of two linear second-order stochastic ordinary differential equations. For this system we obtained the determining equations and the corresponding equivalent transformations which assist with further classifying the system for some selected cases.

Integration Methods for Equations with an Insufficient Number of Lie Point Symmetries

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Lie group methods for determining and using symmetries [1, 2, 3, 4] are a powerful tool for finding exact solutions of systems of nonlinear differential equations and investigating complicated mathematical models. For instance, nth-order ordinary differential equations admitting a n-dimensional solvable symmetry algebra of Lie point symmetries can be integrated by quadratures. If the dimension of the solvable symmetry algebra is $r < n$, then the solution of the original nth-order equation can be reconstructed by r quadratures from the solution of a $(n-r)$th-order reduced equation. If the symmetry algebra is not solvable, it is still possible to reduce the order by r, but, in general, the reconstruction of the solution cannot be carried out by quadratures. If the equation does not admit Lie point symmetries, these procedures cannot longer be applied, and other approaches have to be considered in order to find some exact solution of the equation under study.

There are several mathematical objects related to an ordinary differential equation, or more specifically, to the differential operator associated to the equation, that can be exploited in order to find some exact solution, or some reduction of order, in absence or lack of a sufficient number of Lie point symmetries. The aim of this work is to present some recent ideas and mathematical methods that can be applied to the study of these types of equations, by using one or several of the above mentioned mathematical objects. The procedures involve symmetries more general than Lie point symmetries, including but not limit to higher or generalized symmetries, $\lambda-$symmetries [5], nonlocal symmetries [6], as well as solvable structures [7]. Several examples illustrate how the approaches work in practice.

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Admissible Vectors for a Class of Subgroups of the Symplectic Group

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A class of subgroups of the symplectic group in the form of semi-direct products $M \rtimes_\alpha D$ where M is a vector space is introduced. Thus, these groups have both, a metaplectic and a wavelet repesentation. Criteria on when the two representations will possess equivalent subrepresentations are discussed, allowing one to obtain conditions for the existence of admissible vectors for the metaplectic representation from the well-known results for the wavelet representation. Several examples detailing the admissible vectors are worked out.

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Differential Invariants, Moving Frames, Equivalence and Symmetry, Image Processing, and the Reassembly of Broken Objects

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I will survey the equivariant method of moving frames and how it is used to construct differential invariant signatures for solving equivalence and symmetry problems. Recent applications in image processing, in particular the reassembly of jigsaw puzzles and broken bones in anthropology, will be presented.

Symmetry Analysis and Inflation

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We approach the cosmological inflation through symmetries of differential equations. We consider the general inflaton field in a homogeneous spacetime, and with the use of nonlocal symmetries we are able to write the generic algebraic solution. We use that result in order to generate new inflationary solutions. A series of generalizations of the Chaplygin gas and bulk viscous cosmological solutions for inflationary universes are found. Finally we show how we can construct new inflationary models from already known models by using symmetry transformations.

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Some Submodels of Gas Suspension Equations with Respect to Three Dimensional Subalgebras

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There is considered a system of partial differential equations [1]

$$\frac{d\rho_1}{dt_1} + \rho_1 \text{div } \vec{u}_1 = 0,$$

$$\frac{d\rho_2}{dt_2} + \rho_2 \text{div } \vec{u}_2 = 0,$$

$$\rho_1 \frac{d\vec{u}_1}{dt_1} + m_1 \nabla P(\rho_1, \rho_2) = -\frac{\rho_2}{\tau}(\vec{u}_1 - \vec{u}_2),$$

$$\rho_2 \frac{d\vec{u}_2}{dt_2} + m_2 \nabla P(\rho_1, \rho_2) = \frac{\rho_2}{\tau}(\vec{u}_1 - \vec{u}_2).$$

This system describes isothermal motions in gas suspension. Here ρ_1 and ρ_2 are the partial densities of phases, $P(\rho_1, \rho_2)$ is the pressure of the mixture(a functional parameter), $m_2 = \frac{\rho_2}{\rho_{22}}$ is the volumetric concentration of the second phase, ρ_{22} is the absolute density of the second phase, $m_1 = 1 - m_2$ is the volumetric concentration of the first phase, $\vec{u}_1 = (u_1, v_1, w_1)$ and $\vec{u}_2 = (u_2, v_2, w_2)$ are the velocity vectors of the first and the second phases, $\frac{d}{dt_1} = \frac{\partial}{\partial t} + \vec{u}_1 \cdot \nabla$, $\frac{d}{dt_2} = \frac{\partial}{\partial t} + \vec{u}_2 \cdot \nabla$.

There are discussed 3-dimensional subalgebras of the symmetry algebra of this system [2, 3]. Invariant submodels for all subalgebras, which lead to barochronous motions of two-phase fluid, are obtained and integrated. New invariant solutions of two-phase fluid will be presented.

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Galilean Invariance and Entropy Principle for a System of Balance Laws of Mixture Type

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After defining, in analogy with a mixture of continuous media, a system of balance laws of mixture type, we study the general properties obtained by imposing the Galilean invariance principle. For constitutive equations of local type we study also the entropy principle and we prove the compatibility between the two principles. These general results permit us to construct, from a single constituent theory, the corresponding theory of mixtures in an easy way. As an illustrative example of the general theory, we write down the hyperbolic system of balance laws of mixtures in which each component has 6 fields (mass density, velocity, temperature and dynamic pressure, among which only the last one is a nonequilibrium variable). This is the simplest system after Eulerian mixtures. Global existence of smooth solutions for small initial data is also proved.

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Use of a Solvable Pair of Variational $\mathcal{C}^\infty$-Symmetries to Reduce the Order of Euler–Lagrange Equations

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A method to reduce by four the order of Euler–Lagrange equations associated to nth-order scalar variational problems is presented. The method consists on using a pair of variational $\mathcal{C}^\infty$-symmetries whose commutator satisfies certain solvability condition. After the performed order reduction, a $(2n-2)$-parameter family of solutions for the original Euler–Lagrange equation can be reconstructed by solving two first-order ordinary differential equations.

Applications of Signatures Curves to Characterize Melanomas and Moles

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Noninvasive diagnosis of melanoma persists as a challenge for dermatologists because of the structural differences between moles and melanomas are often indistinguishable to the human eye. Melanoma, the most serious type of skin cancer, develops in the cells that produce melanin - the pigment that gives skin its color. The cancerous skin lesion is capable of spreading throughout the body, making it difficult to treat in advanced cases. In addition, visual similarities between melanoma and mole make diagnosis difficult, and often require the use of an invasive skin biopsy. Dermatologists often use the ABCD method — an acronym for Asymmetry, Border, Color, Diameter — to determine the necessity of a skin biopsy. This research focuses on B and C aspects of the ABCD method.

In this paper we focus on the application of an Euclidean invariant curve, called the *signature curve*, [1], formed by taking curvature and derivative of curvature with respect to arc length of a closed curve, $\Sigma = \{(\kappa(t), \kappa_s(t))\}$, to analyze the contour of melanomas and moles. We calculate the signature curves of the contours of the skin lesions to detect asymmetry, boundary irregularity and diameter size of the skin lesions. By analyzing the signature curves of 60 benign moles and 60 melanomas, we show that the benign and malignant lesions have different global and local symmetry patterns in their signature curves. We will also demonstrate that the regular moles show a high degree of global symmetry, whereas melanomas exhibit multiple types of local symmetry that are embedded within their signature curves. We then turn our attention to the C aspect of the ABCD method by analyzing the color of melanomas and moles. Finally we use ROC Analysis, a key statistical tool, to analyze the performance of our method.

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Group Analysis and Exact Solutions of the Spatially Homogeneous and Isotropic Boltzmann Equation with a Source Term

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The presentation is devoted to group analysis of the spatially homogeneous and isotropic Boltzmann equation with a source term. In fact, the Fourier transform of the Boltzmann equation with respect to the molecular velocity variable is considered. Complete group classification with respect to a source function only depending on independent variables is performed. If a source term includes the dependent variable, then preliminary group classification is given. In the case where the source function also depends on nonlocal term (number of particles), extension of the equivalence Lie group occur. Using these equivalence transformations and preliminary group classification, equations having exact BKW-solutions are derived.

Exact Solutions of a Nonlinear Heat Equation with Maximal Symmetry Algebra

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We consider a nonlinear heat equation which is remarkable from two points of view. Firstly, it admits a maximal 5-dimensional Lie algebra of point symmetries [1] and, secondly, by changing the dependent variable it is reduced to an equation with quadratic nonlinearities possessing an invariant linear subspace of the maximal dimension 5 [2]. For this quadratic equation all exact solutions on the 5-dimensional (polynomial) invariant subspace, both invariant and non-invariant, are constructed in an explicit form.

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Heun's Differential Equation and its q-Deformation

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The hypergeometric differential equation

$$\frac{d^2y}{dz^2} + \frac{\gamma - (\alpha+\beta+1)z}{z(1-z)}\frac{dy}{dz} - \frac{\alpha\beta}{z(1-z)}y = 0 \tag{1}$$

is one of the most important differential equation in mathematics and physics. There are three regular singularities $\{0, 1, \infty\}$ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and it is a standard form of the second order linear differential equation with three regular singularities.

Heun's differential equation is a standard form of the second order linear differential equation with four regular singularities on the Riemann sphere, and it is written as

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-t)}y = 0, \tag{2}$$

with the condition $\gamma + \delta + \epsilon = \alpha + \beta + 1$. The parameter B is an accessory parameter, which is independent of the local exponents, although the hypergeometric differential equation does not have any accessory parameter. Heun's differential equation appears in several systems of physics including general relativity, quantum mechnics and fluid dynamics.

A q-difference analogue of Heun's differential equation was given by Hahn [1], and it is written as

$$\{a_0 + a_1x + a_2x^2\}g(x/q) + \{b_0 + b_1x + b_2x^2\}g(x) + \{c_0 + c_1x + c_2x^2\}g(qx) = 0, \tag{3}$$

where $a_0a_2c_0c_2 \neq 0$. Note that the parameter b_1 in Eq.(3) may be regarded as an accessory parameter. By the limit $q \to 1$, we can obtain Heun's differential equation. Recently the q-Heun equation was recovered ([5]) by two methods, one is by degeneration of Ruijsenaars-van Diejen operator ([4]), and the other is by specialization of the linear q-difference equation related with q-Painlevé VI equation ([2]).

We will explain how we investigate solutions of Heun's differential equation and of the q-Heun equation ([3, 6]) by taking the accessory parameter into account. In particular, we investigate polynomial-type solutions.

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On Models of Dynamics Material Points that are Invariant with Respect to the Poincaré Group

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When building models that are invariant with respect to some group of transformations, we choose the space of dependent and independent variables, the group representation in Cartesian the product of these spaces, the continuation of the group into the space multilinear mappings (on derivatives dependent on independent variables) to a given order, and finally build system of equations, which is an invariant manifold in the continued space [1].

The described method of solving the problem for the model of dynamics material points in the case of Lorentz transformations does not pass, because as dependent variables here you should take the spatial coordinates material points, and as an independent time variable. But in the framework of point transformations a couple of different and simultaneous events relative to the moving reference system will be non-simultaneous, those. It is not possible to build a group representation in the selected space.

In the paper [2] in the framework of the Lee-Backlund transformations manages to get around this problem. Constructing governing equations for such systems. But an example of a specific system succeeded build for one dimensional space only.

In the present work, approximate solutions are constructed. defining equations from [2] with accuracy up to the first order of $|v|/c$. Where v is the speed of a point, c is the speed of light in a vacuum.

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Equation of a Rayleigh Noise Reduction Model for Medical Ultrasound Imaging: Symmetry Classification, Conservation Laws and Invariant Solutions

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Medical ultrasound imaging provides images of internal body for diagnosis. Speckle noise is a major problem degrading image quality. In this presentation, the speckle noise probability distribution is modelled by a Rayleigh type. The variational method is applied to minimize speckle of a noisy image model. Symmetry analysis is used to study the partial differential equation (PDE) obtained. Group classification is performed, invariance solutions and conservation laws of the PDE in cylindrical and Cartesian coordinates are presented. The listed conservation laws and invariant solutions generalize the results previously obtained by various authors.

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Symmetries of the Shallow Water Equations in the Boussinesq Approximation

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The shallow water equations in the Boussinesq approximation are studied in this paper. Two cases of these equations are studied: in Eulerian coordinates and Lagrangian coordinates. A detailed analysis of the admitted Lie groups is given. All invariant solutions of these two representations are presented. Using Noether's theorem, new conservation laws in Eulerian coordinate and Lagrangian coordinates are found.

Conservation Laws in Magnetohydrodynamics and Fluid Dynamics: Lagrangian, Clebsch and Multi-Symplectic Approaches

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This paper describes the use of Lagrangian, Clebsch and Multi-Symplectic approaches to action principles and conservation laws in magnetohydrodynamics (MHD) and fluid dynamics. The connection between Eulerian symmetries and Lagrangian symmetries following the approaches of [2] and [1] and the role of fluid relabeling symmetries are discussed. The use of both Lagrangian and Eulerian symmetries to obtain conservation laws via Noether's theorems are discussed ([1], [3]). Multi-symplectic approaches (e.g. [1]) based on Clebsch variable constrained variational principles are described. Non-local cross helicity and variants of helicity conservation laws for a non-barotropic gas are obtained. Local versions of these helicity conservation laws are obtained for a barotropic gas. The generalized Aharonov Bohm formulation of these nonlocal and local conservation laws are described (e.g. [4], [1]).

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On the Metric Tensor in the External Gravitational Field of an Isolated, Spherically Symmetric, Non-rotating Massive Object

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The Newman-Penrose (”NP”) field equations [1] employ future-oriented null geodesics as coordinate curves. These equations are valid in any subset of a space-time manifold that conforms to the ”spinor space” axioms [2]. Essentially this means that the manifold must be orientable with respect to both space and time [3]. When, additionally, the Ricci scalars vanish, then the initial data for the NP equations is provided by the functional form of the Weyl scalar Ψ_o [4]. This function appears in the first two NP equations together with two unknown metric functions and their second-order derivatives with respect to an affine distance variable r. This pair of equations was originally independently discovered by Sachs [5]. In a flat space-time manifold, affine distance is a particular case of radar distance [6]. In the reference frame of an observer that is in free-fall through the external gravitational field towards the centre of symmetry of a spherically symmetric gravitating object located in an otherwise empty and asymptotically flat space-time manifold, the affine distance variable r is defined along the null geodesic coordinate curves emanating from the observer. The origin of the NP coordinate system is therefore located on the observer, not on the centre of the gravitating object. At first sight the Sachs equations are under-determined, since the two equations contain three unknown dependent variables. However, a Lie group analysis has shown that in the form that the Sachs equations take for the problem under consideration, the pair of equations, as a system, admits a non-trivial Lie symmetry group if, and only if, Ψ_o takes a specific functional form with respect to the variable r [7]. The function Ψ_o then contains four ”constants” (undetermined functions of the remaining variables other than r) and the relevant Lie group is exactly the proper orthochronous Poincaré group, which is the only group of coordinate transformations allowed in a ”spinor space”. This function splits into three equivalence classes according to whether the values of the ”constants” result in Ψ_o admitting two, one or no singularities respectively [8]. In this paper the exact solution of the full set of NP equations is derived for the elements of the metric tensor on the axis along which the observer falls, corresponding to the equivalence class of initial value functions that admit exactly one singularity. The resulting solution is consistent with the Dirac equation for spin 1/2 particles in the sense that a parity inversion is equivalent to a change in the sign of the gravitating mass. Also, the solution contains an expression that, to first degree in v/c, is equal to a Taylor series expansion of the relativistic Doppler factor. This factor results in time dilation and length contraction and produces the well-known relativistic beaming effect with respect to null geodesics emanating from a moving source. The acceleration of the observer relative to the gravitating object, which appears as the integrand in an expression for the speed v, emerges from the solution in a form that is identical to Newton's law in the reference frame of the observer.

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Author Index